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AN

INTRODUCTION

TO THE

DOCTRINE

OF

FLUXIONS.

By John Rowe

Reviſed by ſeveral Gentlemen well ſkill'd in
the MATHEMATICS.

— *Felicibus inde*
Ingeniis aperitur Iter.— CLAUDIAN.

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THE PREFACE.

AS *the Want of a plain and eaſy Introduction, whereby the induſtrious Learner, without the Aſſiſtance of a Maſter, might arrive to a ſmall Degree of Skill in the Doctrine of Fluxions, was the only Inducement to this Work; 'tis hoped the following Sheets will need no Apology; they being principally deſigned to ſupply this Deficiency, and to aſſiſt the ingenious Student, by removing, in ſome Meaſure, thoſe Clouds of Darkneſs and Obſcurity, in which the Treatiſes on this Subject are too often involved.*

Tho' in what is here delivered, neither any of the moſt difficult and abſtruſe Parts of the Doctrine of Fluxions are contained, nor many new Diſcoveries made; yet as it may tend to give the Learner a perfect Notion of, and excite him to ſearch farther into that noble and uſeful Science, it may juſtly deſerve ſome Favour from the Public.

The Entrance on every Science is always the moſt difficult; and the gaining clear and juſt Ideas at firſt, as it is the moſt uneaſy Part, ſo it is the ſureſt Pledge of future Succeſs.

The Mind cannot be eaſily brought to conceive of things entirely unknown before, or in a Manner different from that to which it has hitherto been accuſtomed. Men are uſed to talk and reaſon of finite Quantities only; but here we muſt conceive of thoſe which are leſs than can be aſſigned; of thoſe which are but juſt coming into Being, or juſt vaniſhing out of it.

Here Quantities are conſidered as generated by a continual Increaſe, after the Manner of a Space, which a Thing or Point in Motion deſcribes: Thus a Line is conceived to be generated by a Point in Motion, a Superficies by a Line, *and a Solid by a Superficies. Now as a Line generated by a Point muſt be paſſed over, or deſcribed, with ſome certain Degree of Velocity in the deſcribing Point, in every* where *or* Point *of that Line; ſo any Quantity may be conceived to flow, or increaſe, with a certain Degree of Velocity in every Point of its Deſcription: And if this Velocity be not Uniform, but is either accelerated or retarded, then there will be a certain Degree of Increaſe, peculiar to every Point of the Thing deſcribed. Now the* Velocity *with which any Quantity flows at any particular Point, is what we call the* Fluxion *of that Quantity at that Point.*

This Treatiſe being only deſigned as an Introduction, we have carefully avoided, in the Inverſe Method, the giving of Examples where the Fluents of the Fluxions need Correction: We have alſo avoided, not only in the Inverſe Method, but alſo in the Direct, all thoſe laborious Operations which are more troubleſome than inſtructive: And have endeavour'd to ſteer between an obſcure Brevity on the one Hand, and a tedious Prolixity on the other.

The Queſtions left unanſwered for the Amuſement of the Learner, may be eaſily ſolved by an Application of the Rules here delivered.

The Examples for Illuſtration, are many of them borrowed from Books already wrote on this Subject, and have only to boaſt, that they are ſet here in a different, and perhaps, a clearer Light. The Whole is calculated for the unaſſiſted Learner; and, we flatter ourſelves, is a ſufficient Guide to lead him to the Underſtanding of a compleat, tho' puzzling Syſtem: And if it anſwers this End, 'tis all that is either deſired or expected.

EXETER,
April 30, 1751.

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AN
INTRODUCTION
TO THE
DOCTRINE
OF
FLUXIONS.

CHAP. I.

IN Order to render the *Doctrine of Fluxions* plain and eaſy, we ſhall endeavour to explain the Nature of FLUXION in general, as deliver'd both by Sir *Iſaac Newton* (the great Inventor of *Fluxions)* and by *Leibnitz* and his Followers.

DEFINITION I.

I. FLUXION (according to Sir *Iſaac Newton*) is that whereby a Body in Motion is carried over a given Space in a given Time; that is, *Fluxion* is the ſame as *Velocity.* Thus (Quantities being here conceived to be generated by continual Increaſe, after the Manner of a Space which a Thing or Point in Motion deſcribes; as a Line by a Point, a Superficies by a Line, and a Solid by a Superficies, in Motion:) if we imagine any Point A, to be mov'd along the Line A B the Velocity, or Swiftneſs, with which it moves at any Point of that Line is truly and properly the *Fluxion* of it at that Point.

If this Velocity be uniform thro' the whole Line (*i. e.* if it has no Variation) the Fluxion will be every where the ſame; but if it be any Way accelerated or retarded, then there will be a certain Degree of Velocity or Fluxion peculiar to every Point; and the Velocity wherewith the firſt Velocity is either accelerated or retarded, is called the *Fluxion of the Fluxion*, or the *ſecond Fluxion*: And again, if this Acceleration or Retardation be not uniform, but is increas'd in different Degrees of Proportion in different Parts of the Line; then the Velocity of this

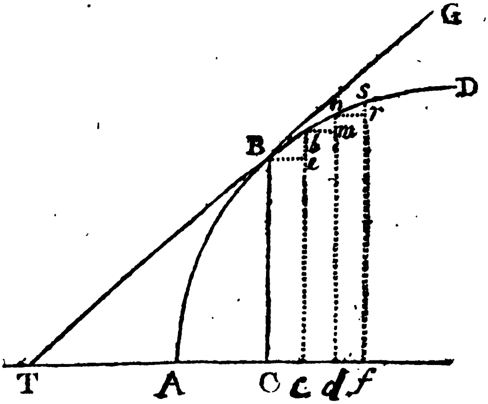
Acceleration or Retardation is called the *Third Fluxion*; and ſo on. This is the juſt and accurate Notion of FLUXION.

But, becauſe in ſome Caſes Velocity may not ſeem ſo particularly concern'd; it may be neceſſary to give the *Foreigners* Definition of it, that nothing be wanting to give the Learner as clear an Inſight as poſſible into the Nature of the Subject; which, as it is highly neceſſary in all Parts of Science, ſo it is particularly in this, where an Obſcurity of Conception in the Outſet may lead him into endleſs Error and Confuſion.

DEFINITION II.

II. Quantities are here ſuppos'd to be generated by a continual Increaſe, as before; and the indefinitely ſmall Particles whereby they are continually increas'd, are call'd the *Fluxions* of thoſe Quantities. Thus, in the Curve ABD, if $b\,c$ be ſuppos'd to be indefinitely near and parallel to the Ordinate B C, and B e parallel to the Abſciſs A C; then B e or C c is call'd the *Fluxion* of the Abſciſs A C, and $e\,b$ the *Fluxion* of the Ordinate C B; and if $n\,d$ be ſuppos'd indefinitely near and parallel to $b\,c$, and $b\,m$ equal and parallel to $c\,d$ or B e; then the Difference between $e\,b$ and $m\,n$ is call'd the *Fluxion* of the *Fluxion* or the *Second Fluxion*; that is, the

Fluxion of *e b* or *Second Fluxion* of C B; and again, if *s f* be ſuppos'd indefinitely near and parallel to *n d*, and *n r* equal and parallel to *d f*



or B *e* or *b m*; then the Difference between the Second Fluxion and *r s*, is call'd the *Third Fluxion* of C B; and ſo on. This is the Notion of FLUXION as deliver'd by *Leibnitz* and his Followers. But theſe Fluxions, we ſhall, in the following Sheets, call by the Names of *Moments, Increments* and *Decrements*; that is, [redacted] *Increments* when the variable Quantities are increaſing, and *Decrements* when they are decreaſing.

3. Now, if the Lines A C and C B could increaſe in an exact Proportion in every Point of the Increments B *e* and *e b*, (as they would do if the Curve B *b* did *exactly* coincide with the Right Line T B G; that is, if the Quantities A C and C B flow'd on with equable or

uniform Motions) it is evident that then the Increments would be accurately as the Fluxions or Velocities; ſince Velocity is always meaſured by the Space uniformly deſcribed in a given Time: But as in Curves, the Proportions of Increaſe or Decreaſe are continually varying, ſo it is evident there muſt be a different Degree of Fluxion or Velocity aſcrib'd to every Point of theſe Increments; and that therefore theſe Increments are not exactly as the Fluxions. However, as the Point *b* is continually nearer to a Coincidence with the Tangent T B G the nearer it approaches the Point of Contact B; ſo if we conceive the Ordinate *c b* to be moved on till it coincide with C B; the very firſt Moment before its Coincidence, the Curve B *b*, and Right-line B G will be infinitely, or rather indefinitely near to a Coincidence with each other; and conſequently, in that Caſe, the Increments B *e* and *e b* will come indefinitely near to meaſure the Ratio of the Fluxions of the Abſciſs and Ordinate A C, and C B, or the Velocities with which they flow at the Point B: or becauſe, if we ſuppoſe the Particles of Time in which any Increments are generated to be indefinitely ſmall, the Acceleration or Retardation of the Fluxion or Velocity with which they are generated will be ſo too; therefore they are indefinitely near in Proportion to the Fluxions of the Quantities of which they are Increments; and therefore (becauſe when any Quantity is increas'd or decreas'd, but by only an infinitely or indefinitely ſmall Particle, that Quantity may be conſider'd as remaining the ſame as it was before;) theſe Increments may be taken as Proportional to, or for the Fluxions in all Operations; and, on the contrary, the Fluxion for the Increment.

4. *Note.* Thoſe Quantities which are ſuppos'd to flow, or be generated by continual Increaſe, are call'd *Fluents*, and *variable* or *flowing* Quantities; and thoſe which admit of no Variation, are call'd, *fixt*, *given*, and *invariable* Quantities. ——The Beginning of the Alphabet, *viz.* a, b, c, d, e, &c. is uſed to expreſs invariable Quantities: and the End, *viz.* z, y, x, &c. variable or flowing Quantities: and the *Fluxions* of thoſe variable Quantities z, y, x, are denoted by $\dot{z}, \dot{y}, \dot{x}$; and their *Second Fluxions*, or the *Fluxions* of $\dot{z}, \dot{y}, \dot{x}$, by $\ddot{z}, \ddot{y}, \ddot{x}$; their *Third Fluxions*, or the *Fluxions* of $\ddot{z}, \ddot{y}, \ddot{x}$, by $\dddot{z}, \dddot{y}, \dddot{x}$, and ſo on: Alſo the *Moment*, *Increment* or *Decrement* of z is denoted by z', of y by y', of x by x'; and the *Second Moments*, *Increments* or *Decrements* of z, y, x, or the *Moments Increments* or *Decrements* of z', y', x', by z'', y'', x'' and ſo on.

The Fluxions and Increments of *a*, *b*, *c*, &c. *viz.* of invariable Quantities are $= 0$.

5. *Note.* Thoſe Fluents which are ſuppos'd to be generated in ye same time ~~to flow together~~, or in equal Times, are call'd Contemporay Fluents; and the Fluxions of theſe Contemporary Fluents are call'd Contemporary Fluxions: So that if two or more of theſe Contemporary Fluents are equal, or in any certain constant Ratio to each other, their Fluxions will be likewiſe equal, or in the ſame Proportion,

BEFORE we proceed to the finding of the Fluxions of Fluents (which is the Buſineſs of the Direct Method of Fluxions,) it may not be improper to premiſe a few Things concerning the new Method of Notation in Algebra.

6. To render the Uſe of Surds as little troubleſome as poſſible, and to accommodate them to fluxional Operations; the Index of the Power to which any Quantity is to be rais'd, is plac'd as the Numerator of a Fraction, whoſe Denominator is the Root to be extracted: Thus the ſquare Root of the Cube of a, which according to the old or common Way of Notation is expreſs'd by $\sqrt{aaa}$, is here expreſs'd $\overline{a}|^{\frac{3}{2}}$ or $a_{\frac{3}{2}}$ and the Cube Root of $ax - xx$, which us'd to be expreſs'd thus $\sqrt[3]{ax - xx}$ is here expreſs'd

thus $\overline{ax-x^2}|^{\frac{1}{3}}$ Alſo the Square Root of x, or $\sqrt{x}$ is thus expreſs'd, $x^{\frac{1}{2}}$

The Reaſon of which is plain if we conſider that the Index or Exponent of the Power of any Quantity, x, is always equal to the Number of the Place that Power bears in a Geometrical Progreſſion whoſe firſt Term and common Multiplier is the Quantity itſelf; or, in other Words, is equal to the Number of Times which the Quantity x muſt be multiplied into itſelf in order to produce that Power: that is, if we take the Geometrical Progreſſion of $1 : x : xx : xxx$ *&c.* and make an Arithmetical Progreſſion of 0. 1. 2. 3 *&c.* then the Numbers in the Arithmetical Progreſſion will be the Indices or Exponents of the correſponding Terms in the Geometrical Progreſſion, and therefore may be conſider'd as the Logarithms of theſe Terms. From hence it follows that the Arithmetical Mean between 1 and 9, for Inſtance, *i. e.* 5, will be the Index of the Geometrical Mean between x or x^1 & x^9 which is x^5. The Arithmetical Mean between 0 and 10 will be $\frac{10}{2}$ or 5 ſo the Geometrical Mean between x^0 or 1 & x^{10} will be $x^{\frac{10}{2}}$ or x^5, that is $\sqrt{x^{10}} = x^5$. Thus likewiſe $\sqrt[3]{x^2}$ will be expreſs'd by $x^{\frac{2}{3}}$ & $\sqrt[7]{\overline{a-x}^4}$ by $\overline{a-x}|^{\frac{4}{7}}$ &c. For the ſame Reaſon alſo if we take a deſcending Series as $xx:x:1:\frac{1}{x}:\frac{1}{x^2}:\frac{1}{x^3}$ *&c.* & take the Arithmetical Series 2. 1. 0.

1.--2.--

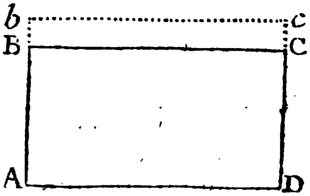
—1.—2.—3. &c. the Number in any Place of the Arithmetical Series will be the Index of the correſponding Place in the Geometrical Series: Thus the Index of $\frac{1}{x^3}$ will be —3, that is $\frac{1}{x^3}$ is $= x^{-3}$. Thus, alſo $\frac{x^2}{x}$ is $= x^{2-1} = x^1$ $\frac{x^1}{x} = x^0 = 1$, $\frac{1}{x^m} = x^{-m}$, and ſo on. In the Exponents of Powers, Addition has the Effect of Multiplication on the reſpective Roots; and Multiplication of Involution, and *e contra*, Subſtraction of Diviſion, and Diviſion of Evolution, or in a Word, Exponents of Powers are entirely Logarithmical with regard to their Roots. So that $\frac{x^{\frac{2}{3}}}{x^{\frac{1}{2}}}$ is $= x^{\frac{2}{3}-\frac{1}{2}} = x^{\frac{1}{6}}$, $\overline{x^{a+x}}|^{3}$ is $= x^{3a+3x}$, $\overline{x^{a+x}}|^{\frac{1}{3}}$ is $= x^{\frac{a+x}{3}}$. And $\frac{x^m}{x^n}$ is $= x^{m-n}$. And $\frac{\overline{x^{m+n}}|^{a}}{x^{\frac{m}{a}}}$ is $= x^{am+an-\frac{m}{a}}$.

If what has been ſaid, be duly conſider'd, no Difficulty in the new Notation will occur; and the Knowledge of it is abſolutely neceſſary in order to underſtand the following Chapters.

CHAP. II.

Of finding the FLUXION *of a given* FLUENT.

7. RULE 1. To find the Fluxion of a ſimple Fluent wherein there is but one variable Quantity. Mark the variable Quantity, or flowing Term with a Dot over it: Thus the Fluxion of ax is $a\dot{x}$. For,



Let A D=B C=a, and A B = D C = x, and let $b\,c$ be ſuppos'd equal and indefinitely near and parallel to BC. *i. e.* let $Bb = Cc = x'$. Then will $CB \times Bb = ax'$ be the Moment of the Rectangle AC or ax; and becauſe $\dot{x}$ may be ſubſtituted for x' (by *Art.* 3.) $a\dot{x}$ will be = its Fluxion.

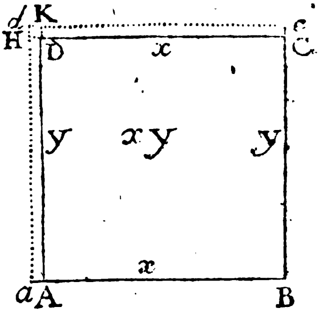
8. RULE 2. To find the Fluxion of a Quantity compounded of different ſimple Quantities or Fluents connected together with the Signs + and —. Find the Fluxion of each ſimple Fluent by Rule 1*ſt*, then connect theſe Fluxions together with the Signs of their reſpective Fluents, and you will have the Fluxion of the Quantity requir'd. Thus the Fluxion of $a^2+ax-by$ is $0+a\dot{x}-b\dot{y}$. For ſince x and y flow or increaſe together; when ax is

increas'd by its Moment ax'; that is, when ax becomes $ax+ax'$; $-by$ will become $-by-by'$; and a^2 will remain the ſame, as being invariable, and conſequently its Moment will be $=0$: therefore, then the Quantity $a_2+ax-by$ with its indefinitely ſmall Increaſe or Decreaſe will become $a^2+ax-by+ax'-by'$ and the Increaſe or Decreaſe alone will be $= ax'-by'$. Wherefore, becauſe x' and y' are expreſſive of $\dot{x}$ and $\dot{y}$ (*Art.* 3.) $a\dot{x}-b\dot{y}$ is $=$ the Fluxion of $a^2+ax-by$.

Again, the Fluxion of $x-y+cz$ is $=\dot{x}-\dot{y}+c\dot{z}$. And the Fluxion of a^2x+bz is $= a^2\dot{x}+b\dot{z}$.

9. RULE 3. To find the Fluxion of the Product of two or more Quantities drawn into each other. Multiply the Fluxion of each Quantity ſeparately by the other, or Product of the Reſt; and the Sum of theſe Products will be the Fluxion requir'd. Thus the Fluxion of xy is $\dot{x}y+x\dot{y}$, and the Fluxion of axy is $a\dot{x}y+ax\dot{y}$: Alſo the Fluxion of xyz is $\dot{x}yz+x\dot{y}z+xy\dot{z}$; and the Fluxion of $axyz$ is $a\dot{x}yz+ax\dot{y}z+axy\dot{z}$.

1. That the Fluxion of xy is $\dot{x}y+x\dot{y}$ may be thus demonſtrated:



Let the Rectangle $AC=xy$ be a Fluent or flowing Quantity, *i. e.* let it increaſe by a continual Enlargement of its Dimention, or Sides $AB=x$ and $BC=y$. Now in order to repreſent this increaſe in its naſcent State, let the prick'd Lines *ad* and *dc* be drawn very near and parallel to AD and DC; and let CD and AD be continued on to H and K; then A*a* or DH and C*c* or DK will repreſent the Increments of the Lines CD and AD, and *adc*CDA the Moment or Increment of the whole Rectangle AC. Now if $DK=y'$ and $DH=x'$, the whole increaſed Rectangle will be $=xy+xy'+x'y+x'y'$ and conſequently the whole Increment or Moment is $=xy'+x'y+x'y'$: But in the very firſt Moment of the Exiſtence of this Increment, or juſt as it is coming into Being, or beginning to be, $x'y'$ bears no aſſignable Ratio to either $x'y$ or xy'. (for as $x'y' : x'y :: y' : y$ and y' by Suppoſition is infinitely leſs than y, *ergo*, &c.) and therefore may very well be expunged or rejected: So that $x'y+xy'$ will expreſs the Moment or Increment of the above Rectangle as ſoon as it begins to be; and then only it is that the Increment

crement is expreſſive of the Fluxion, or of the Velocity with which the Rectangle flows ~~at~~, or increases ~~the Point D~~: Hence by ſubſtituting the Fluxion for the Increment (*ſee Art.* 3.) we have $\dot{x}y+x\dot{y}=$ the Fluxion of the Rectangle xy.

2. Much after the ſame Manner it may be demonſtrated, that the Fluxion of axy is $a\dot{x}y+ax\dot{y}$. Thus let x' expreſs the very firſt Moment of x, or the Increaſe of x juſt as it begins to be, and y' of y. Now when x becomes $x+x'$, y will become $y+y'$, but a will remain the ſame as being invariable; and therefore axy with indefinitely ſmall increaſe will become $=a\times\overline{x+x'}\times\overline{y+y'}=axy+ax'y+axy'+ax'y'$ and its increaſe alone $=ax'y+axy'+ax'y'$: But as $ax'y:ax'y'::y:y'$; and therefore, becauſe y is infinitely greater than y', $ax'y$ is infinitely greater than $ax'y'$: So that (becauſe when any Quantity is increaſed but by an indefinitely ſmall Particle, that Quantity may be conſidered as remaining the ſame as before) the above Increment or Increaſe of the Fluent axy, may very well be conſidered as $=ax'y+axy'$, and by ſubſtituting the Fluxion for the Increment (*Art.* 3.) we have its Fluxion $=a\dot{x}y+ax\dot{y}$.

3. Alſo that the Fluxion of xyz is $\dot{x}yz+x\dot{y}z+xy\dot{z}$ may be proved after the ſame Manner; thus, when x flows and becomes $x+x'$, then y and z will flow, and become equal to $y+y'$ and

$z+z'$; and therefore then the Fluent xyz will become $=\overline{x+x'}\times\overline{y+y'}\times\overline{z+z'}=xyz+x'yz+xy'z+xyz'+x'y'z+xy'z'+x'yz'+x'y'z'$ from which ſubtracting the ſaid Fluent xyz, we ſhall have its Moment or Increment alone $=x'yz+xy'z+xyz'+x'y'z+xy'z'+x'yz'+x'y'z'$ which may very juſtly be conſidered and taken as $=x'yz+xy'z+xyz'$ ſince the Ratio of this to $x'y'z+xy'z'+x'yz'+'x'y'z$ is indefinitely great, or greater than any that can be aſſigned, as may be proved thus, $x'yz : x'y'z :: y : y'$, and $xy'z : xy'z' :: z : z'$, and $xyz' : x'yz' :: x : x'$, and $x'yz : x'y'z' :: yz : y'z'$; therefore, (becauſe y is infinitely greater than y', z than z', x than x', and yz than $y'z'$) $x'yz$ is infinitely greater than $x'y'z$ and $x'y'z'$, $xy'z$ than $x'y'z$, and xyz' than $x'yz'$; and therefore &c. wherefore, becauſe (*Art.* 3.) x' is expreſſive of $\dot{x}$, and y' of $\dot{y}$ and z' of $\dot{z}$, by ſubſtituting the Fluxion for the Increment, we have the Fluxion of $xyz=\dot{x}yz+x\dot{y}z+xy\dot{z}$. Or thus, ſubſtitute v for xy; then by Demonſt. 1. will $\dot{v}=\dot{x}y+x\dot{y}$, and the Fluxion of $xyz=\dot{v}z+v\dot{z}$; therefore, by reſtitution or writing $\dot{x}y+x\dot{y}$ inſtead of $\dot{v}$, and xy inſtead of v, we ſhall have $\dot{x}yz+x\dot{y}z+xy\dot{z}=$ the Fluxion of xyz, as before.

4. And thus, alſo it may be proved, that the Fluxion of $axyz$ is $=a\dot{x}yz+ax\dot{y}z+axy\dot{z}$; for ſubſtituting v for axy we ſhall have by Dem. 2.

$\dot{v}=a\dot{x}y+ax\dot{y}$, and the Fluxion of $vz=\dot{v}z+v\dot{z}=$ the Fluxion of $axyz$, which by Reſtitution is $=a\dot{x}yz+ax\dot{y}z+axy\dot{z}$.

Hence,

10. The Fluxion of xx is $=\dot{x}x+x\dot{x}$, the Fluxion of xxx is $=\dot{x}xx+x\dot{x}x+xx\dot{x}$; that is, the Fluxion of x^2 is $=2x\dot{x}$, the Fluxion of x^3 is $3x^2\dot{x}$; and if we put $m=1,2,3$, or any poſitive Number whatſoever, the Fluxion of x^m will be $mx^{m-1}\dot{x}$.

11. RULE 4. To find the Fluxion of a Fraction. Multiply the Fluxion of the Numerator into the Denominator, from the Product of which ſubtract the Fluxion of the Denominature drawn into the Numerator; then this Remainder divide by the Square of the Denominator, and you will have the Fluxion ſought. Thus the Fluxion of $\frac{ax}{y}$ is $=\frac{a\dot{x}y-\dot{y}ax}{y^2}$; For make $z=\frac{ax}{y}$; then will $yz=ax$, and therefore the Fluxion of $yz=$ that of ax (*Art.* 5.) that is, $\dot{y}z+y\dot{z}=a\dot{x}$; wherefore by Tranſpoſition and Diviſion $\dot{z}=\frac{a\dot{x}-\dot{y}z}{y}$ *i. e.* (by reſtitution or writing $\frac{ax}{y}$ for z) $\dot{z}=\frac{a\dot{x}y-ax\dot{y}}{y^2}=$ the Fluxi-

on of z or $\frac{ax}{y}$. Alſo the Fluxion of $\frac{x^3}{y^2}$ is $= \frac{3x^2\dot{x}y^2 - 2y\dot{y}x^3}{y^4}$; the Truth of which, may be demonſtrated as before, by making $z = \frac{x^3}{y^2}$: Again the Fluxion of $\frac{1}{x}$ is $= \frac{-\dot{x}}{x^2}$; the Fluxion of $\frac{1}{x^2}$ is $= \frac{-2x\dot{x}}{x^4}$; the Fluxion of $\frac{1}{x^3}$ is $= \frac{-3x^2\dot{x}}{x^6}$, &c. But by the new Notation (*See Art.* 6.) $\frac{1}{x}$ is $= x^{-1}$; $\frac{1}{x^2}$ is $= x^{-2}$; $\frac{1}{x^3}$ is $= x^{-3}$; and $\frac{-\dot{x}}{x^2}$ is $= -x^{-2}\dot{x}$; $\frac{-2x\dot{x}}{x^4}$ or $\frac{-2\dot{x}}{x^3}$ is $= -2x^{-3}\dot{x}$; $\frac{-3x^2\dot{x}}{x^6}$ or $\frac{-3\dot{x}}{x^4}$ is $= -3x^{-4}\dot{x}$, &c. Hence therefore,

12. The Fluxion of x^{-1} is $= -x^{-2}\dot{x}$; the Fluxion of x^{-2} is $= -2x^{-3}\dot{x}$; the Fluxion of x^{-3} is $= -3x^{-4}\dot{x}$, &c. and if m repreſent any Negative Number whatſoever, the Fluxion of x^m will be $= mx^{m-1}\dot{x}$.

13. RULE 5. To find the Fluxion of any Power of a Fluent or flowing Quantity. Multiply the whole Term by the Index of the Power, then ſubtract 1 from the ſaid Index, and multiply this whole Term into the Fluxi-

on of the Root of the Fluent; and you will have the Fluxion of the Fluent required. Thus the Fluxion of $\overline{x+x^2}|^3$ is $=3\times\overline{x+x^2}|^{3-1}\times\overline{\dot{x}+2x\dot{x}}$ *i. e.* $=3\times\overline{x+x^2}|^2\times\overline{\dot{x}+2x\dot{x}}$; the Fluxion of $\overline{2ax-x^2}|^{\frac{1}{2}}$ is $=\frac{1}{2}\times\overline{2ax-x^2}|^{-\frac{1}{2}}\times\overline{2a\dot{x}-2x\dot{x}}$, which is $=\frac{1}{2}\times\frac{1}{\sqrt{2ax-x^2}}\times\overline{2a\dot{x}-2x\dot{x}}$ *i. e.* $=\frac{a\dot{x}-x\dot{x}}{\sqrt{2ax-x^2}}$; the Fluxion of $\overline{a+ax-x^2}|^{\frac{2}{3}}$ is $=\frac{2}{3}\times\overline{a+ax-x^2}|^{-\frac{1}{3}}\times\overline{0+a\dot{x}-2x\dot{x}}$ *i. e.* $=\frac{2}{3}\times\frac{1}{\sqrt[3]{a+ax-x^2}}\times\overline{a\dot{x}-2x\dot{x}}$ or $\frac{2a\dot{x}-4x\dot{x}}{3\times\sqrt[3]{a+ax-x^2}}$: And, univerſally, the Fluxion of $x^{\frac{m}{n}}$ is $=\frac{m}{n}x^{\frac{m}{n}-1}\dot{x}$; where m or n, or both, may be either Affirmative or Negative whole Numbers or Fractions; ſo that the Index or Exponent $\frac{m}{n}$ expreſſes or repreſents any Affirmative or Negative whole Number or Fraction whatſoever. For by *Art.* 10 and 12. if m repreſents any Affirmative or Negative whole Number, the Fluxion of x^m will be $mx^{m-1}\dot{x}$; therefore, if we ſuppoſe $y=x^{\frac{m}{n}}$, or which is the ſame, $y^n=x^m$, (n being any

Affirmative or Negative whole Number likewiſe) by *Art.* 5. we have $ny^{n-1}\dot{y} = mx^{m-1}\dot{x}$ which divided by ny^{n-1} gives $\dot{y} = \frac{mx^{m-1}\dot{x}}{ny^{n-1}}$ *i. e.* by writing $x^{\frac{m}{n}}$ for its equal y, $\dot{y} = \frac{mx^{m-1}\dot{x}}{n \times \overline{x^{\frac{m}{n}}}\Big|^{n-1}}$ $= \frac{m}{n} \times \frac{x^{m-1}\dot{x}}{x^{\frac{mn-m}{n}}} = \frac{m}{n} x^{\frac{m}{n}-1} \dot{x}$ which is = the Fluxion of $x^{\frac{m}{n}}$ above; therefore, &c. And that either m or n, or both, may repreſent any Fractions as well as whole Numbers is plain, ſince this $\frac{m}{n}$ expreſſes the Quotient of any whole Number divided by another, and may be taken for a new m or n; and ſo on, *ad infinitum.*

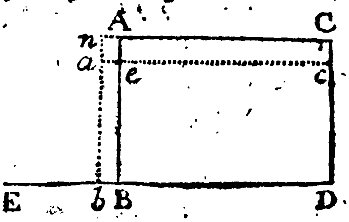
14. RULE 6. To find the Fluxion of a Logarithm. The Fluxion of the Hyperbolic Logarithm of any Quantity, is equal to the Fluxion of that Quantity divided by the Quantity itſelf: Thus the Fluxion of the hyperbolic Logarithm of x is $= \frac{\dot{x}}{x}$; the Truth of which we ſhall hereafter Demonſtrate (*See* Chap. 14.

Queſt. 10.) then, as the hyp. Log. of 10 (*viz.* 2.302585092 9 &c.) : is to the common Log. of 10 (*viz.* 1) : : ſo is the hyp. Log. of any Number x : to the common Log. of that ſame Number; that is, if we put L=2.30258 &c. As L : 1 : : hyp. Log. of x : common Log. of x, and therefore, (*Art.* 5.) : : Fluxion of hyp. Log. of x : Fluxion of common Log. of x. Hence we have $L : 1 :: \frac{\dot{x}}{x} : \frac{\dot{x}}{Lx}$ = the Fluxion of the common Log. of x; or, becauſe $\frac{1}{L}=0.4342944819$ &c. if we put this $\frac{1}{L}=M$, then the Fluxion of the common Log. of x, *viz.* $\frac{\dot{x}}{Lx}$, will be $=\frac{\dot{x}}{x}\times M$, *i.e.* the Fluxion of the hyperbolic Logarithm of any Number multiplied by (M or) 0.434294 &c. gives the Fluxion of the common Logarithm of that ſame Number.

15. *Note,* It is ſometimes expedient, in order to find the Fluxion, to divide the Equation given, by one or more of the variable Quantities contained in moſt of the Terms: And to ſubſtitute ſingle Letters for compound Quantities. Thus if the given Equation were

$xy+a-axy+y^2=yz$, divide both Sides of the Equation by y; then $x+\frac{a}{y}-ax+y=z$; and the Fluxion of this Equation is $\dot{x}-\frac{a\dot{y}}{y^2}-a\dot{x}+\dot{y}=\dot{z}$. And if the Equation given were $x-by+\frac{x^2y}{z}=az$, ſubſtitute v for $\frac{x^2y}{z}$; then $x-by+v=az$, and this in Fluxions is $\dot{x}-b\dot{y}+\dot{v}=a\dot{z}$.

16. *Note*, Hitherto we have ſuppoſed, when one variable Quantity in any Fluent increaſes, that the others, if any, increaſe likewiſe; but it ſometimes happens, that ſome of them decreaſe while the others increaſe: in which Caſe, the Fluxions of theſe decreaſing are Negative, with reſpect to thoſe of the increaſing Quantities; and therefore the Signs of the Terms which are affected with them, ought to be Negative. Thus, if whilſt x increaſes, y decreaſes, the Fluxion of the Fluent xy will be $=\dot{x}y-x\dot{y}$. To prove which, let the Line A B be ſuppoſed perpendicular to the Line DE and equal and parallel to the Line C D and to move along



on the ſaid Line DE towards E; and at the ſame Time, let the Point A be ſuppoſed to move along the Line AB towards B, ſo as that when the Point B is advanced to b, the Point A may be arrived at a (ab and ac being ſuppoſed indefinitely near and parallel to AB and AC;) then the little Parallelogram A C ce ſubtracted from the other Beab will be the Moment of the Rectangle ABDC. Now if we put AC or BD$=x$, AB or CD$=y$, Increment Bb or $ea=x'$, and decrement Ae or $na=y'$; then will $e\mathrm{B}=y-y'$, and the little Rectangle $\mathrm{B}a=\overline{y-y'}\times x'=yx'-x'y'$, *i. e.* = the little Rectangle Bn, leſs the little Rectangle en; but ſince theſe two little Rectangles are to each other as BA to eA, and the Point e is ſuppoſed to be indefinitely near to the Point A, or Ae to be but juſt beginning to be, or coming into Being; therefore the little Rectangle Bn cannot properly be ſaid to be diminiſhed or leſſened by the ſubtracting from it the little Rectangle en which is ſo infinitely leſs: Wherefore the ſaid little Rectangle Ba may be conſidered as equal to yx'; from which if we ſubtract the little Rectangle Ac, which is $=xy'$, we ſhall have $yx'-xy'=$ the Moment of the Rectangle xy or BC; therefore by ſubſtituting $\dot{x}$ and $\dot{y}$ for x' and y' (*Art.* 3.) we ſhall have the Velocity with which the ſaid Rectangle increaſes or decreaſes [redacted], or the Fluxion of it ~~at that Point~~ $=y\dot{x}-x\dot{y}$. Again, if z decreaſes whilſt y and x increaſes, then the Fluxion of the Fluent xyz will be $=\dot{x}yz+x\dot{y}z-xy\dot{z}$. Underſtand the ſame of the Fluxion of any other Fluent of a like Nature whatſoever.

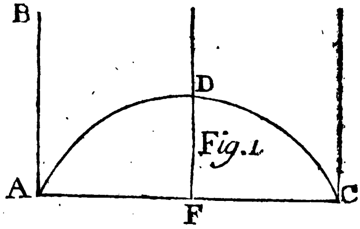
If what has been already delivered, be thoroughly underſtood, we hope the Learner will meet with but few Difficulties in the Application thereof; to which we now proceed.

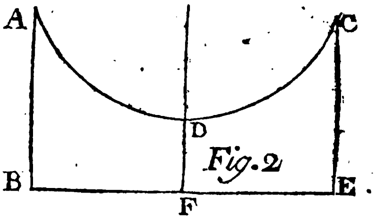
CHAP. III.

Of finding the Maxima *and* Minima (*or greateſt and leaſt*) *of variable* Quantities.

17. IF a variable Quantity is of ſuch a Nature that it increaſes continually without end, or decreaſes till it vaniſhes, its greateſt or leaſt Magnitude cannot be aſſigned or determined; but if there is a certain Limit beyond which it cannot paſs, and it is determinable when it arrives at that Limit; or, if at firſt it increaſes till a certain aſſignable Term and then decreaſes, or firſt decreaſes, and then increaſes; its Magnitude at ſuch Term, is called a *Maxi-*

mum or *Minimum*: Now every ſuch *Maximum* or *Minimum*, muſt be in its own Nature a ſtable or invariable Quantity; and therefore, becauſe an invariable Quantity has no Fluxion, its Fluxion is equal to nothing.





18. To illuſtrate and make this plainer: Let the right Line B A be ſuppoſed perpendicular to the right Line AC (*Fig.* 1.) or BE (*Fig.* 2.) and to be carried along the ſaid Line with any given Velocity, and in ſuch a Manner, as that it always be parallel to its firſt Situation; and, at the ſame time, let a Point be conceived to move along upon the Line A B, from A, with ſuch Velocity, as that it be always in the Curve A D C: Then will the Velocity with which the Line B A moves along the Line A C, or B E, be the Fluxion of the Abſciſſa; the Velocity with which the Point moves along the Line A B, be the Fluxion of the Ordinate; and the Velocity with which the Point moves along the Curve, be the

Fluxion of the Curve. Now when the Point, moving from A towards B, arrives at D, it can proceed no father, but muſt return towards A again; becauſe the Ordinate F D (*Fig.* 1.) is the greateſt, or (*Fig.* 2.) the leaſt poſſible, *i. e.* before the Point arrives at D, the Ordinate (*Fig.* 1.) is continually increaſing, and afterwards continually decreaſing, or (*Fig.* 2.) is continually decreaſing, and afterwards, continually increaſing; therefore at that Point, it is neither increaſing nor decreaſing, but is invariable, and therefore its Fluxion is $=0$.

19. When there are two or more variable Quantities in an Expreſſion, repreſenting the Value of a *Maximum* or *Minimum*, the Value of thoſe Quantities may be determined by ſuppoſing them to flow ſeparately, or one by one, whilſt the reſt are conſidered as invariable. For Inſtance, if it were required to find the greateſt Parallelopipedon, that can be contained under a given Superficies, it is evident that each of its Sides muſt ſeparately flow out to a certain determinate Point, beyond which it muſt not paſs; and therefore its Fluxion at that Point muſt ceaſe, or become $=0$; or, becauſe if the Fluxion of the given Expreſſion of the *Maximum* or *Minimum*, when only one of the Quantities is conſidered as variable,

be not $=0$, the ſame Expreſſion may become greater or leſs without varying the Values of thoſe which are conſidered as conſtant; therefore, when it is the greateſt or leaſt poſſible, each of thoſe Fluxions muſt then become $=0$.

20. *Note*, 1. In curvilineal Spaces, it is in effect the ſame Thing to ſeek the greateſt Area that can be contained under a given Perimiter, as to ſeek a given Area under the leaſt Perimiter: And the ſame holds good, in reſpect of Solids and their Surfaces.

Note, 2. If any Quantity is a *Maximum* or *Minimum*, all the Powers of it will be ſo too; as will alſo the Product or Quotient, ariſing by its being multiplied or divided by any invariable or given Quantity. Thus, if $\frac{ab}{c} \times \overline{x^2 - 2ax}|^{\frac{1}{3}}$ repreſent a *Maximum* or *Minimum*, then $x^2 - 2ax$ will be alſo a *Maximum* or *Minimum*, and conſequently its Fluxion muſt be $=0$.

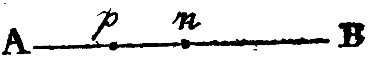
Note 3. Generally, when a variable Quantity admits of a *Maximum*, its *Minimum* is nothing; and when it admits of a *Minimum*, its *Maximum* is infinite.

EXAMPLE I.

21. *To find the Value of x, in Terms of a and b, when $bx-ax^2$ is a* Maximum.

BECAUSE the Fluxion of a *Maximum* is $=0$, therefore the Fluxion of $bx-ax^2$ muſt be $=0$; that is $b\dot{x}-2ax\dot{x}=0$. Now if we divide this Equation by $\dot{x}$, we ſhall have $b-2ax=0$; therefore, by Tranſpoſition, we have $2ax=b$ and by Diviſion $x=\frac{b}{2a}$ which is the Value of x required.

EXAMPLE II.



22. *To find the Point p in the given right Line* AB, *where the Rectangle* Ap *into* pB *is a* Maximum, *or greater than any other Rectangle* An *into* nB.

PUT the given right Line AB$=a$, and Ap $=x$; then pB$=a-x$, and by the Data A$p\times p$B$=x\times\overline{a-x}=ax-x^2=$ a *Maximum*; therefore the Fluxion of $ax-x^2$ is $=0$, that is, $a\dot{x}-2x\dot{x}=0$, which divided by $\dot{x}$ gives $a-2x=0$, therefore $2x=a$, and $x=\frac{1}{2}a$. Conſequently the Rectangle of the Parts Ap and pB are the greateſt when thoſe Parts are equal.

Or thus, Put A$p=x$, and pB$=y$; then A$p\times p$B$=xy=$ a *Maximum*: Now it being evident that if x increaſe, y muſt decreaſe, therefore (*Art.* 16.) $\dot{x}$ and $\dot{y}$ are Negative to each other; and conſequently the Fluxion of xy, when a *Maximum*, is $\dot{x}y-x\dot{y}=0$: But it is likewiſe evident, that the Increment of x is equal to the Decrement of y, or $\dot{x}=\dot{y}$; therefore ſtrike both out in the above Fluxion of xy; then we ſhall have $y-x=0$, and therefore $x=y$, as before.

EXAMPLE III.

23. *To find the Point p in the given right Line* AB, *when* $\overline{\mathrm{A}p}|^m\times\overline{p\mathrm{B}}|^n$ *is a* Maximum.

PUT the given right Line AB$=a$, and A$p=x$; then pB$=a-x$, and $\overline{\mathrm{A}p}|^m\times\overline{p\mathrm{B}}|^n=x^m\times\overline{a-x}|^n=$ a *Maximum*, therefore its Fluxion is $=0$; that is, (*ſee Art.* 13. and 9.) $mx^{m-1}\dot{x}\times\overline{a-x}|^n+n.\overline{a-x}|^{n-1}\times-\dot{x}\times x^m=0$ (*i. e.*) $mx^{m-1}\dot{x}\times\overline{a-x}|^n-n\overline{a-x}|^{n-1}\dot{x}\times x^m=0$, which divided by $x^{m-1}\dot{x}$ gives $m\overline{a-x}|^n-n\overline{a-x}|^{n-1}x=0$; therefore $m.\overline{a-x}|^n=n.\overline{a-x}|^{1-1}x$; then divide by $\overline{a-x}|^{n-1}$ and it will be $m\,.\,\overline{a-x}=nx$, *i. e.* $ma-mx=nx$, or $mx+nx=ma$, therefore $x=\frac{m}{m+n}a$, and the Point p is determined.

Or thus, Put A$p=x$, and pB$=y$; then $y^n x^m$ = a *Maximum*, which in Fluxions is $ny^{n-1}\dot{y}x^m - mx^{m-1}\dot{x}y^n=0$, (*ſee Art.* 16.) but $\dot{x}$ and $\dot{y}$ are equal (*Art.* 22.) therefore throw both $\dot{x}$ and $\dot{y}$ out then $ny^{n-1}x^m - mx^{m-1}y^n=0$. And $ny^{n-1}x^m= mx^{m-1}y^n$. Now by dividing both Sides of this Equation by $y^{n-1}x^{m-1}$ we ſhall have $nx=my$ therefore $m : n :: x : y$: So that the Segments are in direct Proportion to the Powers in the *Maximum*.

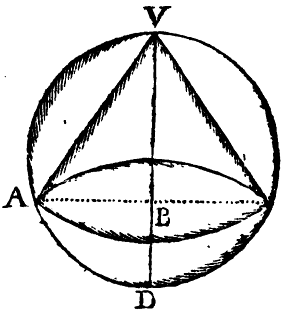
EXAMPLE IV.

24. *To find the Values of x, y, and z, when $x+y+z=a$ and $xy+z^2=$ a* Minimum.

BECAUSE $x+y+z=a$, therefore $x=a-y-z$, which ſubſtituted for x, gives $xy+z^2= ay-y^2-yz+z^2$ = a *Minimum.* Now by *Art.* 19. we may conſider y as variable whilſt z remains invariable, and z as variable whilſt y remains invariable; therefore, by ſuppoſing z invariable, the Fluxion of the above Expreſſion of the *Minimum* will be $a\dot{y}-2y\dot{y}-\dot{y}z=0$, and by ſuppoſing y invariable, it will be $-y\dot{z}+2z\dot{z}=0$: Now by dividing the Firſt of theſe two fluxional Equations by $\dot{y}$ we have $a-2y-z=0 \because z=a-2y$, and by dividing the Second of them by $\dot{z}$ we have $-y+2z=0 \because z=\frac{1}{2}y$. Hence $\frac{1}{2}y=a-2y$, and therefore $y=\frac{2}{5}a$, and conſequently $z=(\frac{1}{2}y=)\,\frac{1}{5}a$, and $x=(a-y-z=)\,\frac{2}{5}a$.

EXAMPLE V.

25. *To find a Cone inſcribed in a given Sphere, whoſe convex Surface ſhall be a* Maximum.



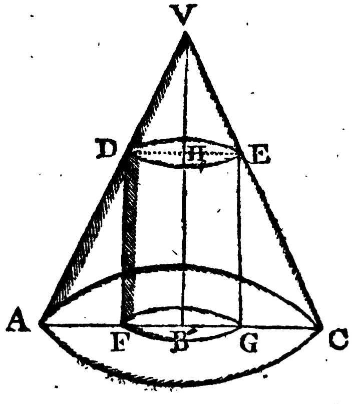
PUT the Diameter of the given Sphere (V D) $=a$, the Altitude of the inſcribed Cone (V B) $=x$, and 3.14159 &c. $=c$; then $\mathrm{BD}=a-x$. Now, by a well known Property of Circles, $\mathrm{VB}\times\mathrm{BD}=\overline{\mathrm{BA}}|^{2}$; that is, $x\times\overline{a-x}=ax-x^{2}=\overline{\mathrm{BA}}|^{2}$: And by 47 *E.* 1. $\mathrm{AV}=\overline{\overline{\mathrm{VB}}|^{2}+\overline{\mathrm{BA}}|^{2}}|^{\frac{1}{2}}$, *i. e.* $\mathrm{AV}=\overline{x^{2}+ax-x^{2}}|^{\frac{1}{2}}=\overline{ax}|^{\frac{1}{2}}$. But $\mathrm{BA}\times c=\overline{ax-x^{2}}|^{\frac{1}{2}}\times c=\overline{c^{2}ax-c^{2}x^{2}}|^{\frac{1}{2}}=\frac{1}{2}$ the Circumference of the Cone's Baſe, which drawn into its ſlant Height $(\mathrm{AV}=)\;\overline{ax}|^{\frac{1}{2}}$, gives $\overline{c^{2}ax-c^{2}x^{2}}|^{\frac{1}{2}}$

$\times \overline{ax}|^{\frac{1}{2}} = \overline{c^2a^2x^2 - c^2ax^3}|^{\frac{1}{2}} =$ its Curve Superficies, which by Queſt. muſt be a *Maximum*; therefore $c^2a^2x^2 - c^2ax^3 =$ the Square of a *Maximum*, or a *Maximum* alſo (*Art.* 20, *Note* 2.) and its Fluxion, therefore $2c^2a^2x\dot{x} - 3c^2ax^2\dot{x} = 0$, which divided by $c^2ax\dot{x}$ gives $2a - 3x = 0$; therefore $3x = 2a$ and $x = \frac{2}{3}a$. So that the Convex Surface of the Cone will be the greateſt, when its Altitude is $= \frac{2}{3}$ of the Sphere's Diameter.

26. *N. B.* There was no need of introducing c in the above, for VA × AB = a *Maximum*; the Radii of Circles being as their Circumferences. And therefore in ſuch like Caſes, we ſhall omit inſerting it as often as may be in the following Examples.

EXAMPLE VI.

27. *To find the greateſt Cylinder that can be inſcribed in a given right Cone* e. g. *whoſe Altitude* BV *is* $= a$, *and Baſe-diameter* AC $= b$.

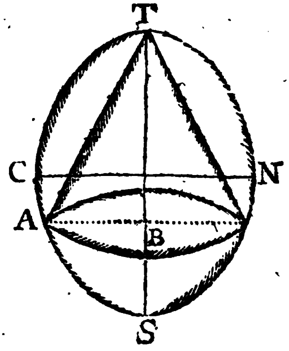


PUT the Diameter of the Cylinder

DE or FG$=x$; then, the Δ's VAC and VDE being Similar, as AC : BV :: DE : HV, *i. e.* $b : a :: x : \frac{ax}{b}=$HV ∵ HB=BV—HV$=a-\frac{ax}{b}=$ the Altitude of the Cylinder. Whence (*ſee Art.* 26.) $\overline{FG}^2 \times BH = x^2 \times \overline{a-\frac{ax}{b}} = \frac{abx^2-ax^3}{b} =$ a *Maximum*. Conſequently (*ſee Art.* 20. *Note* 2.) $bx^2-x^3=$ a *Maximum*, in Fluxions $2bx\dot{x}-3x^2\dot{x}=0$, which divided by $x\dot{x}$ gives $2b-3x=0$ ∵ $3x=2b$ and $x=\frac{2}{3}b=$FG or DE. Conſequently BH ($=a-\frac{ax}{b}$) $=\frac{1}{3}a$. So that the Diameter and Altitude of the Cylinder, will be to the Diameter and Altitude of the Cone, as $\frac{2}{3}$ to 1, and as $\frac{1}{3}$ to 1, when the Cylinder is the greateſt poſſible.

EXAMPLE VII.

28. *To find the largeſt Cone that can be inſcribed in a given prolate Spheroid,* viz. *whoſe Tranſverſe and Conjugate Diameters* TS *and* CN *are a and b.*



PUT the Altitude of

the inſcribed Cone B T$=x$, then B S$=$T S$-$T B$=a-x$. Now by a well known Property of Ellipſes $\overline{TS}|^2 : \overline{CN}|^2 :: TB \times BS : \overline{AB}|^2$ *i. e.* $a^2 : b^2 :: x \times \overline{a-x} : \frac{ab^2x-b^2x^2}{a^2} = \overline{BA}|^2$ which (*Art.* 26.) drawn into ($\frac{1}{3}$B T) $\frac{1}{3}x$ gives $\frac{ab^2x^2-b^2x^3}{3a^2} =$ a *Maximum.* Conſequently, (*Art.* 20. *Note* 2.) $ax^2-x^3=$ a *Maximum*, which in Fluxions is $2ax\dot{x}-3x^2\dot{x}=0$; then divide by $x\dot{x}$ and it gives $2a-3x=0$. Conſequently $3x=2a$ and $x=\frac{2}{3}a$, that is, the Altitude of the Cone muſt be $=\frac{2}{3}$ of the Diameter of the Spheroid.

EXAMPLE VIII.

29. *To find the internal Dimenſions of a cylindrical Cup, whoſe Capacity is given* $=a$; *when made with the leaſt poſſible Quantity of Silver of a given thickneſs.*

PUT the Diameter $=x$, and .78539 &c. $=c$; then $cx^2=$ the Area of the Bottom, and therefore $\frac{a}{cx^2}=$ the Altitude: But $4cx$ being $=$ the Circumference of the Bottom, therefore $4cx \times \frac{a}{cx^2} = \frac{4a}{x} =$ the inſide Curve Surper-

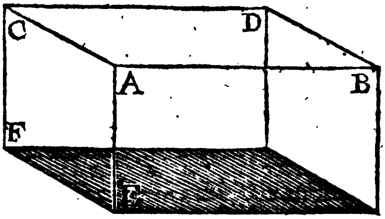
fices.

ficies*; which, becauſe the Quantity of Silver is the leaſt poſſible, is a *Minimum*. And this therefore in Fluxions is $2cx\dot{x} - \frac{4a\dot{x}}{x^2} = 0$, which multiplied by x^2 gives $2cx^3\dot{x} - 4a\dot{x} = 0$, and this divided by $2\dot{x}$ gives $cx^3 - 2a = 0$; therefore $cx^3 = 2a$, and $x = \overline{\left.\frac{2a}{c}\right|}^{\frac{1}{3}}$.

\#add cx^2 hence $cx^2 + \frac{4a}{x} =$ whole minimo, superficies

EXAMPLE IX.

30. *To find the internal Dimenſions of a Ciſtern, in the Form of a rectanglar Solid, i. e, whoſe Bottom and all four Sides are rectangular; when its Capacity is $= a$ and made with the leaſt poſſible Quantity of Lead of a given thickneſs.*



PUT the inſide Length A B or C D x, Breadth A C or B D $= y$, and Depth A E or C F $= z$; then $xyz = a$, and therefore $x = \frac{a}{yz}$. Now the inſide Superficies of the Bottom and four Sides is $= \text{CD} \times \text{DB} + 2\text{BA} \times \text{AE} + 2\text{CA} \times \text{AE}$, *i. e.*

$=xy+2xz+2yz$ (or, by ſubſtituting $\frac{a}{yz}$ for x,) $=\frac{a}{z}+\frac{2a}{y}+2yz$, which, becauſe the Ciſtern is made of the leaſt poſſible Quantity of Lead, is equal a *Minimum*; and this therefore, by making y and z flow ſeparately (*ſee Art.* 19.) in Fluxions is $2\dot{y}z-\frac{2a\dot{y}}{y^2}=0$, and $2y\dot{z}-\frac{a\dot{z}}{z^2}=0$; the firſt of which Equations multiplied by y^2 gives $2\dot{y}zy^2-2a\dot{y}=0$, and this divided by $2\dot{y}$ gives $zy^2-a=0 \therefore z=\frac{a}{y^2}$: And the ſecond Equation multiplied by z^2 gives $2y\dot{z}z^2-a\dot{z}=0$, and this divided by $\dot{z}$ is $2yz^2-a=0 \therefore z=\left.\overline{\frac{a}{2y}}\right|^{\frac{1}{2}}$. Hence we have $\left.\overline{\frac{a}{2y}}\right|^{\frac{1}{2}}=\frac{a}{y^2}$ and this ſquared is $\frac{a}{2y}=\frac{a^2}{y^4}$ which multiplied by $2y^4$ gives $ay^3=2a^2$, therefore $y^3=2a$ and $y=\left.\overline{2a}\right|^{\frac{1}{3}}$. Whence $z\ (=\frac{a}{y^2})=\frac{a}{\left.\overline{2a}\right|^{\frac{2}{3}}}=\frac{2a}{2\times\left.\overline{2a}\right|^{\frac{2}{3}}}=\frac{1}{2}\times\left.\overline{2a}\right|^{\frac{1}{3}}$. And $x\ (=\frac{a}{yz})=\frac{a}{\left.\overline{2a}\right|^{\frac{1}{3}}\times\frac{1}{2}\left.\overline{2a}\right|^{\frac{1}{3}}}=\frac{a}{\frac{1}{2}\times\left.\overline{2a}\right|^{\frac{2}{3}}}=\left.\overline{2a}\right|^{\frac{1}{3}}$. So that $x=y=2z$, that is, the

Length and Breadth will be equal, and each equal to twice the Depth.

Or thus, Let the inſide Length $= x$, Breadth $= y$, and Depth $= z$; then the Superficies of the Bottom and four Sides $= xy + 2xz + 2yz =$ a *Minimum*, as before; or, by ſubſtituting $-b$ for x, it will be $= -by - 2bz + 2yz$. In Fluxions $-b\dot{y} - 2b\dot{z} + 2\dot{y}z + 2y\dot{z} = 0$. And taking the homologous Terms * $2\dot{y}z - b\dot{y} = 0$, and $2y\dot{z} - 2b\dot{z} = 0$; by Diviſion and Tranſpoſition $2z = b$, and $y = b$, therefore, by Reſtitution $x = y = 2z$. But $xyz = a$, *i. e.* $\frac{1}{2}x^3$, or $\frac{1}{2}y^3 = a \because x$ or $y = \overline{2a}|^{\frac{1}{3}}$ and $z = \frac{1}{2}\overline{2a}|^{\frac{1}{3}} = \overline{\frac{1}{4}a}|^{\frac{1}{3}}$.

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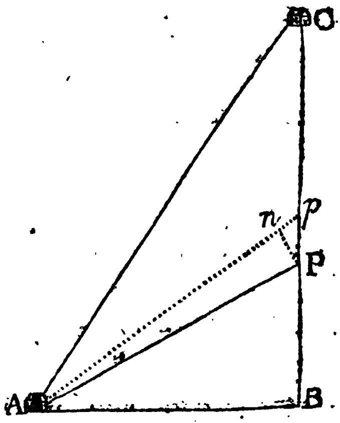
* This in Effect is the ſame as making y and x flow ſeparately according to *Art.* 19.

EXAMPLE X.

31. *A Gentleman wants to ride from the City* A *to the City* C, *the Cities being* a *Miles apart; now from* A *to* B, *which is* b *Miles, or from* A *to any Place* P *in the Road* C B *which is perpendicular to the Road* B A



he can ride after the Rate of c *Miles an hour; but from* B *to* C *he can ride faſter, or after the rate of* d *Miles an hour: To find* P, *the Place to which he muſt directly ride in order to perform his Journey in the leaſt poſſible Time.*

By 47 *E.* 1. BC$=\overline{\overline{CA}|^2-\overline{AB}|^2}|^{\frac{1}{2}}=\overline{a^2-b^2}|^{\frac{1}{2}}$ which put $=e$; alſo let BP $=x$, then PC $=e-x$; therefore by Queſt. $\frac{e-x}{d}=$ the Number of Hours he will be riding from P to C. Again by 47 *E.* 1. AP$=\overline{\overline{PB}|^2+\overline{BA}|^2}|^{\frac{1}{2}}=\overline{x^2+b^2}|^{\frac{1}{2}}$, therefore by Queſt. $\frac{\overline{x^2+b^2}|^{\frac{1}{2}}}{c}=$ the Number of Hours he will be riding from A to P. Hence we have $\frac{\overline{x^2+b^2}|^{\frac{1}{2}}}{c}+\frac{e-x}{d}=$ the Number of Hours he will be performing his Journey, which by Queſt. muſt be a *Minimum:* Therefore in Fluxions $\frac{\frac{1}{2}\times\overline{x^2+b^2}|^{-\frac{1}{2}}\times 2x\dot{x}}{c}-\frac{\dot{x}}{d}=0$, that is, $\frac{\frac{1}{2}\times\frac{1}{\overline{x^2+b^2}|^{\frac{1}{2}}}\times 2x\dot{x}}{c}-\frac{\dot{x}}{d}=0$, or $\frac{x\dot{x}}{c\times\overline{x^2+b^2}|^{\frac{1}{2}}}-\frac{\dot{x}}{d}=0$, then multiply by $c\times\overline{x^2+b^2}|^{\frac{1}{2}}\times d$, and we ſhall have $dx\dot{x}-c\times$

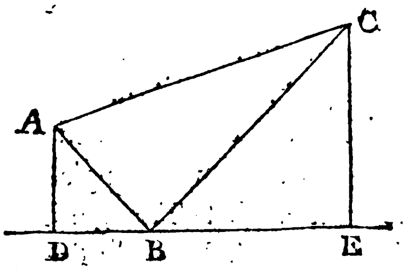
$\overline{x^2+b^2}|^{\frac{1}{2}}\dot{x}=0$, and this tranſpoſed and divided by $\dot{x}$ gives $dx=c\times\overline{x^2+b^2}|^{\frac{1}{2}}$ which ſquared is $d^2x^2=c^2x^2+c^2b^2$, therefore $d^2x^2-c^2x^2=c^2b^2$ and $x^2=\frac{c^2b^2}{d^2-c^2}$ conſequently $x=\frac{cb}{\overline{d^2-c^2}|^{\frac{1}{2}}}=$ BP.

Or thus, Let Ap be ſuppoſed indefinitely near to A P, and the little circular Arch Pn deſcribed with the Radius A P; that is, let Pp expreſs the indefinitely ſmall Increment of B P and np that of A P. Now it is evident, if P be the Place to which he muſt directly ride, that the Increments Pp and np muſt be as d to c, ſince then only can they be paſſed over in the ſame time; but the little Triangle Pnp is Similar to the right angled Triangle A B P; (for the Arch Pn being indefinitely ſmall, may be conſidered as a little right Line perpendicular to Ap; and ſo the Angle PAp being indefinitely little, the Angle ApB or npP may be conſidered as equal to the Angle A P B, and therefore the Angle pPn as equal to the Angle PAB.) Conſequently as $d:c::$ A P : P B: But when the Hypothenuſe and Perpendicular are d and c, the Baſe by 47 *E.* 1. will be $\overline{d^2-c^2}|^{\frac{1}{2}}$, therefore, $\overline{d^2-c^2}|^{\frac{1}{2}}:(\text{AB})\,b::c:\frac{bc}{\overline{d^2-c^2}|^{\frac{1}{2}}}=$ B P, as before.

EXAMPLE XI.

32. *Let the Triangle* ABC *have one Angle* B *in the right Line* DE. *To find a* Maximum *of the Sum of its Perpendiculars* AD *and* CE *dropt from the other two Angles on the right Line aforeſaid.*

N. B. $AB=3=a$, $BC=4=b$, and the $\angle ABC=90°$.



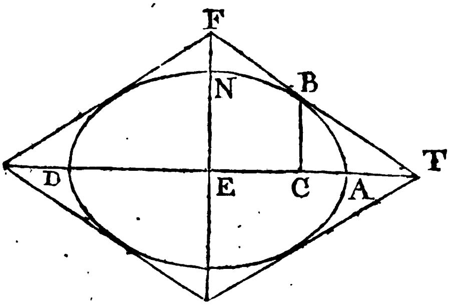
PUT $DB=x$; then by 47 *E.* 1. $AD=\overline{\overline{AB}|^2-\overline{BD}|^2}\big|^{\frac{1}{2}}=\overline{a^2-x^2}\big|^{\frac{1}{2}}$. Now, becauſe the $\angle ABC$ is right, the $\angle CBE$ is the Complement of the $\angle ABD$, and therefore is $=\angle BAD$; conſequently the Triangles ABD and CBE are Similar; therefore $AB : BD :: BC : CE$, *i. e.* $a : x :: b : \frac{bx}{a}=CE$. Hence we have $AD+CE=\overline{a^2-x^2}\big|^{\frac{1}{2}}+\frac{bx}{a}=$ a *Maximum*

mum by Queſt. in Fluxions $\frac{1}{2}\times\overline{a^2-x^2}\big|^{-\frac{1}{2}}\times -2x\dot{x}+\frac{b\dot{x}}{a}=0$, *i. e.* $-\frac{x\dot{x}}{\overline{a^2-x^2}\big|^{\frac{1}{2}}}+\frac{b\dot{x}}{a}=0$, which multiplied by $\overline{a^2-x^2}\big|^{\frac{1}{2}}\times a$ gives $-ax\dot{x}+\overline{a^2-x^2}\big|^{\frac{1}{2}}b\dot{x}=0$: Therefore by Tranſpoſition and dividing by $\dot{x}$ we have $ax=\overline{a^2-x^2}\big|^{\frac{1}{2}}b$ by Involution $a^2x^2=a^2b^2-b^2x^2$. Conſequently $a^2x^2+b^2x^2=a^2b^2$, and $x^2=\frac{a^2b^2}{a^2+b^2}$ therefore $x=\frac{ab}{\overline{a^2+b^2}\big|^{\frac{1}{2}}}=2.4$. Hence AD+CE $\left(=\overline{a^2-x^2}\big|^{\frac{1}{2}}+\frac{bx}{a}\right)=1{,}8+3.2=5=$ the Sum of the Perpendiculars, when that Sum is the greateſt poſſible.

EXAMPLE XII.

33. *To find the greateſt Ellipſis that can be inſcribed in a Rhombus, whoſe Diagonals are given.*

Let the Rhombus be drawn, and the Ellipſis inſcribed as in the Fig. and from the Point of Contact B the Ordinate BC let fall. Put $ET=a$, $EF=b$, and EA or $ED=x$. Then (ſee *De L'Hoſpital's* Conic Sections *Art.* 57.)



57.) as $ET : EA :: EA : EC$, *i. e.* $a : x :: x : \frac{x^2}{a} = EC \because CA = x - \frac{x^2}{a} = \frac{ax - x^2}{a}$, $CD = \frac{ax + x^2}{a}$, and $CT = \frac{a^2 - x^2}{a}$. By Sim. $\triangle$s $TE : EF :: TC : CB$, *i. e.* $a : b :: \frac{a^2 - x^2}{a} : \frac{a^2b - bx^2}{a^2} = CB$. Now by a well known Property of Ellipſes $DC \times CA : \overline{CB}|^2 :: DE \times EA : EN^2$, *i. e.* $\left(\frac{ax + x^2}{a} \times \frac{ax - x^2}{a} =\right) \frac{a^2x^2 - x^4}{a^2} : \left(\overline{\frac{a^2b - bx^2}{a^2}}\Big|^2 =\right)$ $\frac{a^4b^2 - 2a^2b^2x^2 + b^2x^4}{a^4} :: x^2 :$ $\frac{a^6b^2x^2 - 2a^4b^2x^4 + a^2b^2x^6}{a^6x^2 - a^4x^4} = b^2 - \frac{b^2x^2}{a^2} = \overline{EN}|^2$. Now, becauſe the Ellipſis is to be the greateſt poſſible (*Art.* 26.) $\overline{EN}|^2 \times \overline{EA}|^2 = \overline{b^2 - \frac{b^2x^2}{a^2}} \times x^2 = b^2x^2 - \frac{b^2x^4}{a^2} =$ the Square of a *Maximum*, which therefore divided by b^2 (*ſee Art.* 20. *Note* 2.) gives $x^2 - \frac{x^4}{a^2} =$ a *Maximum*, which in Fluxions is $2x\dot{x} - \frac{4x^3\dot{x}}{a^2} = 0$, and this multiplied by a^2 and divided by $2x\dot{x}$ gives $a^2 - 2x^2 = 0$ ∵ $2x^2 = a^2$ and $x = a\sqrt{\frac{1}{2}}$,

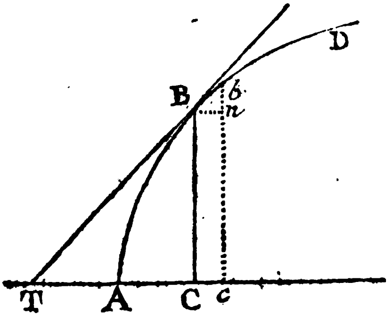
COROLLARY.

The tranſverſe Axis of the greateſt Ellipſis that can be inſcribed in a Rhombus is equal the greateſt Diagonal drawn into $\sqrt{\frac{1}{2}}$, and the conjugate Axis equal the leaſt Diagonal drawn into $\sqrt{\frac{1}{2}}$, that is, the Axes are to the Diagonals, as $\sqrt{\frac{1}{2}}$ to 1; which is the ſame Proportion, as the Diameter of a Circle bears to the Diagonal of its circumſcribing Square.

CHAP. IV.

Of drawing TANGENTS *to* CURVES.

34. AS it is the Nature of all Curves, to have their Directions alter'd in every Point, ſo the Directions of every Curve will be various as the indefinite Number of Particles or Increments, of which it is compoſed; and to find the Direction of a Curve in any given Point, or a general Expreſſion for its Direction in all Points, is what is meant by drawing a Tangent to it.



If the right Line T B be drawn to coincide with any Point B of the geometrical Curve A B D ſo as to touch, but not to cut it, that Line is a Tangent to the Curve in that Point: And becauſe no two Lines can coincide, unleſs they have the ſame Direction, therefore the

Direction of the Tangent, is properly, the Direction of the Curve in the Point of Contact: And what is required, by drawing a Tangent, is generally to find the Sub-tangent C T, or the Diſtance from the Point C, of the Ordinate B C to the Point T, where the Axis produced, if occaſion, is interſected by the Tangent.

35. Let *c b* be ſuppoſed parallel to C B, and B *n* equal and parallel to C *c*; then if the Line *c b* be removed towards C B, in a parallel Motion, till it coincides with it, the Moment before its coincidence, the Triangle B *n b* will be in its evaneſcent State; or, which is the ſame thing, if the ſaid two Lines be ſeparated from their Coincidence, the very firſt Moment of their Separation, produces the ſaid Triangle in its naſcent State; and in that Moment, the Line *n b*, terminated by the Line B *n* at one End, and by the Curve at the other, comes indefinitely near to the Tangent T B produced: Conſequently the Triangles *b n* B and B C T come indefinitely near to Similarity, and may be conſidered as Similar: Wherefore, then, *bn* : *n* B : : BC : CT, or, (putting AC $= x$, and CB $= y$) $y' : x' :: y : \frac{x'y}{y'} =$ C T, that $\dot{y} : \dot{x} ::$

$y : \frac{\dot{x}y}{\dot{y}} = CT$. And this is a general Expreſſion for the Subtangent of every Curve whoſe Abſciſſa is x and Ordinate y.——Now by means of the Fluxion of the Equation of the Curve, we may get the Value of $\dot{x}$ expreſs'd in Terms that will be all affected with $\dot{y}$, which therefore being multiplied by $\frac{y}{\dot{y}}$ will give the Subtangent CT in definitive or known Terms freed from Fluxions; by which the ſought Tangent BT to a given Point may be drawn.

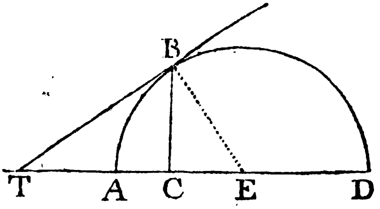
36. *Note*, When $\dot{x}$ and $\dot{y}$ are Negative to each other, (*Art.* 16.) the general Expreſſion ($\frac{\dot{x}y}{\dot{y}}$ for the Subtangent, will be $-\frac{\dot{x}y}{\dot{y}}$, which ſhews that it lies on the other ſide of the Ordinate, with regard to the Abſciſſa x: For tho' $-\frac{\dot{x}y}{\dot{y}}$ may be, and really is a negative algebraic Quantity, yet it may alſo repreſent a geometrical Quantity, which is always affirmative; and as every Subtangent is in its own Nature poſitive, therefore the negative Sign (either in the general or definitive Expreſſion) only ſhews where we

muſt look for the Subtangent; that is, whether x the Abſciſs, or ſome Part of it be (as in the Affirmative it always is,) or be not (as in the Negative it never is,) included in the Expreſſion for the Subtangent.

There are other Methods of drawing Tangents, but this is the moſt general and eaſy.

EXAMPLE I.

37. *To draw a Tangent to a Circle.*



Put the Abſciſs AC$=x$, Ordinate CB$=y$, and Radius EA or ED$=r$; then CD$=2r-x$. Now by the Property of Circles AC × CD$=$CB^2, *i. e.* $2rx-x^2=y^2$, and this Equation put into Fluxions is $2r\dot{x}-2x\dot{x}=2y\dot{y}$, which divided by $2r-2x$ gives $\dot{x}=\frac{2y\dot{y}}{2r-2x}=\frac{y\dot{y}}{r-x}$, and this multiplied by $\frac{y}{\dot{y}}$ (*Art.* 35.) or ſubſtituted for $\dot{x}$ in $\left(\frac{\dot{x}y}{\dot{y}}\right)$ the general Expreſſion for the Subtangent, gives the Subtangent CT $(=\frac{\dot{x}y}{\dot{y}})=\frac{y^2}{r-x}$ = (by putting $2rx-x^2$ for its equal y^2 as above) $\frac{2rx-x^2}{r-x}=\frac{\overline{CB}|^2}{EC}$. Wherefore if the Diſtance ſignified by this Expreſſion be ſet off from the Point C in the Dimeter D A produced, we ſhall have the Point T to which the Tangent may be drawn.

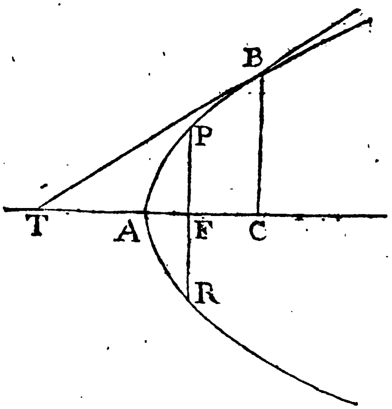
N. B. The Subtangent above, may be found otherwiſe, by the Similarity of Triangles only: For the ∠E B T being a right one, the right angled Triangles E C B and B C T will be Similar, and therefore E C : C B : : B C : C T$=\frac{CB^2}{EC}$.

EXAMPLE II.

38. *To draw a Tangent to the Apollonian or common Parabola.*

Suppoſe F to be the Focus; and P R the Parameter, which put $=a$; alſo, put the Abſciſſa A C $=x$, and ordinate C B $=y$; B being the Point to which the Tangent T B is required to be drawn. Then, as is well known, by the Nature of the Curve PR × AC$=\overline{CB}|^2$, that is, $ax=y^2$. Now by putting both Sides

of



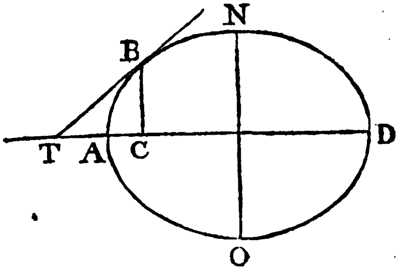
this Equation into Fluxions, we ſhall have $a\dot{x} = 2y\dot{y}$, therefore $\dot{x} = \frac{2y\dot{y}}{a}$ which ſubſtituted for $\dot{x}$, gives the general Expreſſion for the Subtangent CT, *viz.* $\frac{\dot{x}y}{\dot{y}}$ (*ſee Art.* 35.) $= \frac{2y^2}{a} =$ (by ſubſtituting for y^2, its equal ax above.) $\frac{2ax}{a} = 2x$. Whence it appears, that the Subtangent CT is double the Abſciſſa AC, and conſequently AT is = AC.

39. But to draw Tangents to all Sorts of Parabolas univerſally; let the Abſciſſa $= x$, ordinate $= y$, and the Parameter $= 1$. Then the Equation will be $x = y^m$, and in Fluxions $\dot{x} = my^{m-1}\dot{y}$, which ſubſtituted for $\dot{x}$ gives the

Subtangent $\frac{\dot{x}y}{\dot{y}} = my^m =$ (by putting x for y^m its value,) mx, ſo that when $m = 2$, it will be $=$ $2x$ as before.

EXAMPLE III.

40. *To draw a Tangent to an Ellipſis.*

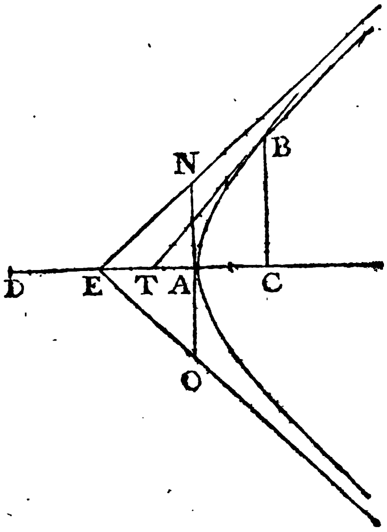


Put the tranſverſe Diameter $AD = t$, Conjugate $NO = c$, Abſciſſa $AC = x$, and ordinate $CB = y$. Now, by the Nature of the Curve, $\overline{AD}|^2 : \overline{NO}|^2 :: AC \times CD : \overline{CB}|^2$, *i, e.* $t^2 : c^2 :: x \times \overline{t-x} : \frac{c^2}{t^2}\overline{tx-x^2} = y^2$, and by putting each Side of this Equation into Fluxions we ſhall have $\frac{c^2}{t^2}\overline{t\dot{x}-2x\dot{x}} = 2y\dot{y}$ and this divided by $\frac{c^2}{t^2}\overline{t-2x}$ gives $\dot{x} = \frac{2t^2y\dot{y}}{c^2t-2c^2x}$

which ſubſtituted for $\dot{x}$ in $\frac{\dot{x}y}{\dot{y}}$ the general Expreſſion for the Subtangent (*ſee Art.* 35), gives the Subtangent CT $= \frac{2t^2y^2}{c^2t - 2c^2x} =$ (by writing $\frac{c^2}{t^2}\overline{tx - x^2}$ for y^2 its equal) $\frac{2c^2t^3x - 2c^2t^2x^2}{c^2t^3 - 2c^2t^2x}$ $= \frac{2tx - 2x^2}{t - 2x}$. Whence we may obſerve that AT ($=$ CT $-$ CA $= \frac{2tx - 2x^2}{t - 2x} - x) = \frac{tx}{t - 2x}$ $=$ that Part of the Subtangent which falls without the curve.

EXAMPLE IV.

41. *To draw a Tangent to an Hyperbola.*

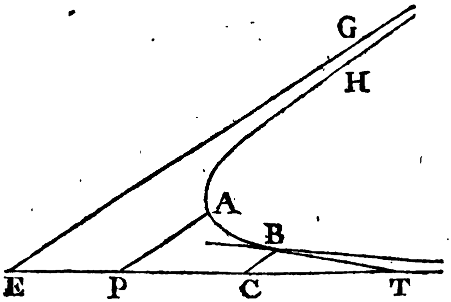


Put the tranſverſe Diameter D A $= t$, Conjugate NO $= c$, Abſciſſa AC $= x$, and Ordinate CB $= y$. Now, by the Nature of the Curve, as $\overline{DA}|^2 : \overline{NO}|^2 :: DC \times AC : \overline{CB}|^2$ *i. e.* $t^2 \; c^2 :: \overline{t+x} \times x : \frac{c^2}{t^2}\overline{tx+x^2} = y^2$, and by putting both Sides of this Equation of the Curve into Fluxions, we ſhall have $\frac{c^2}{t^2}\overline{t\dot{x}+2x\dot{x}} = 2y\dot{y}$ therefore, by Diviſion, $\dot{x} = \frac{2t^2y\dot{y}}{c^2t+2c^2x}$ which ſubſtituted for $\dot{x}$ in $\frac{\dot{x}y}{\dot{y}}$ or multiplied by $\frac{y}{\dot{y}}$ (*Art.* 35.) gives the Subtangent C T $= \frac{2t^2y^2}{c^2t+2c^2x}$ (which by writing for y^2, its equal $\frac{c^2}{t^2}\overline{tx+x^2}$ is) $= \frac{2tx+2x^2}{t+2x}$. So that, AT, that Part of the Subtangent without the Curve, is $= \frac{2tx+2x^2}{t+2x} - x = \frac{tx}{t+2x}$.

EXAMPLE V.

42. *To draw a Tangent to an Hyperbola between its Aſſymptotes; that is, taking one of its Aſſymptotes for an Axis.*

Let

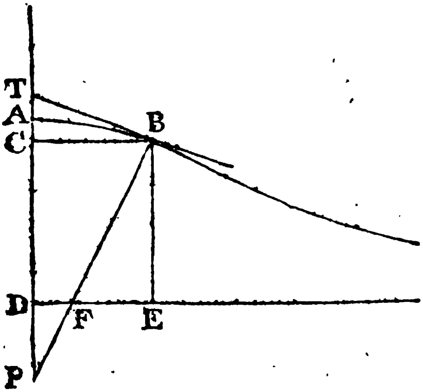


Let E G and E T be the Aſſymptotes of the Hyperbola H A B, whoſe Vertex is A, draw A P and B C parallel to the Aſſymptote E G: then will A P be the Parameter, and equal to PE, which put $= a$; E C an Abſciſs, which put $= x$; and C B an Ordinate, which put $= y$. Now, becauſe when x increaſes, y decreaſes, $\dot{x}$ and $\dot{y}$ are Negative to each other, and the general Expreſſion for the Subtangent is $-\frac{\dot{x}y}{\dot{y}}$ which ſhews that the Point T lies on the other Side of the Ordinate C B with regard to E (*Art.* 36.) By the property of the Curve E C : E P :: PA : C B, *i. e.* $x : a :: a : y$ $\therefore$ $x = \frac{a^2}{y}$ which in Fluxions is $\dot{x} = \frac{-a^2\dot{y}}{y^2}$ and this ſubſtituted for $\dot{x}$ in $\frac{\dot{x}y}{\dot{y}}$ gives the

Subtangent CT $=\frac{a^2\dot{y}y}{y^2\dot{y}}=\frac{a^2}{y}=$ (by writing xy for a^2) $\frac{xy}{y}=x$, ſo that CT muſt be equal to CE.

EXAMPLE VI.

43. *To draw a Tangent to the Conchoid of* Nicomedes *.



Let fall the Perpendicular BE on the Aſſymptote DE and draw BC equal and parallel to ED. Put PD $=a$, DA $=b=$ FB, DC $=x=$ EB, and CB $=y=$ DE. Then by 47 *E.* 1. $\overline{\overline{FB}|^2-\overline{BE}|^2}|^{\frac{1}{2}}=$ EF, *i. e.* $\overline{b^2-x^2}|^{\frac{1}{2}}=$ EF. But the △'s CPB and BEF are Similar, therefore BE : EF :: PC : CB, *i. e.* $x:\overline{b^2-x^2}|^{\frac{1}{2}}::a$

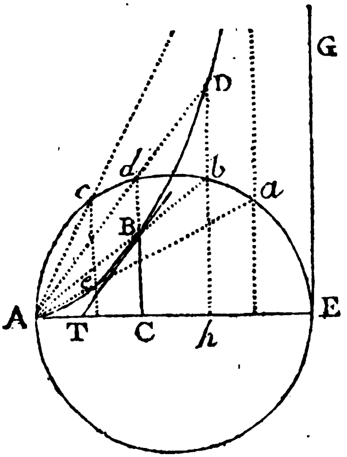
$+x$

$+x : y = \frac{a+x}{x} \times \overline{b^2-x^2}\big|^{\frac{1}{2}}$ which is the Equation of the Curve; and this in Fluxions is $\dot{y} = \frac{x\dot{x}-\overline{a-x}\,\dot{x}}{x^2} \times \overline{b^2-x^2}\big|^{\frac{1}{2}} - \frac{1}{2} \times \overline{b^2-x^2}\big|^{-\frac{1}{2}} 2x\dot{x} \times \frac{a+x}{x} = \frac{-a\dot{x}}{x^2} \times \overline{b^2-x^2}\big|^{\frac{1}{2}} - \frac{x\dot{x}}{\overline{b^2-x^2}\big|^{\frac{1}{2}}} \times \frac{a+x}{x} = \frac{-ab^2\dot{x}-x^3\dot{x}}{x^2 \times \overline{b^2-x^2}\big|^{\frac{1}{2}}}$ which ſubſtituted for $\dot{y}$ in $-\frac{\dot{x}y}{\dot{y}}$, the general Expreſſion for the Subtangent when $\dot{x}$ and $\dot{y}$ are Negative to each other (*Art.* 36.) gives the Subtangent C T $= \frac{yx^2 \times \overline{b^2-x^2}\big|^{\frac{1}{2}}}{ab^2+x^3}$ (which by ſubſtituting for y its equal in the above Equation of the Curve, is) $= \frac{\overline{a+x} \times \overline{b^2-x^2} \times x^2}{ab^2x+x^4} = \frac{\overline{a+x} \times \overline{b^2x-x^3}}{ab^2+x^3}$.

* This Curve is thus generated; from a fixt Point P, which is called the Pole of the Conchoid, let any Number of right Lines PA, PB, be drawn cutting the right Line D E, which is an Aſſymptote to the Curve; and let the Diſtances D A, F B, be made equal to each other, and a Line drawn thro' the Points A, B; then will this Line be a Curve, called by its Inventor *Nicomedes*, a Conchoid.

EXAMPLE VII.

44. *To draw a Tangent to the Ciſſoid of* Diocles *.



Let A B D be the Ciſſoid, whoſe generating Circle is A F E and Aſſymptote E G. Put the Diameter $AE = a$, Abſciſs $AC = x$, and Ordinate $CB = y$. Now by the Nature of the Curve $dC : CA :: AC : CB$ (For ſince by the Generation of the Curve, the Arches Eb and Ad muſt be equal, therefore $bb = dC$ and $bE = CA$; and becauſe the $\triangle$'s bbA and BCA are alike, therefore $Ab : bb :: AC : CB$; but by the Property of Circles $Ab : bb :: bb : bE \because bb : bE :: AC : CB$, *i. e.* $dC : CA$

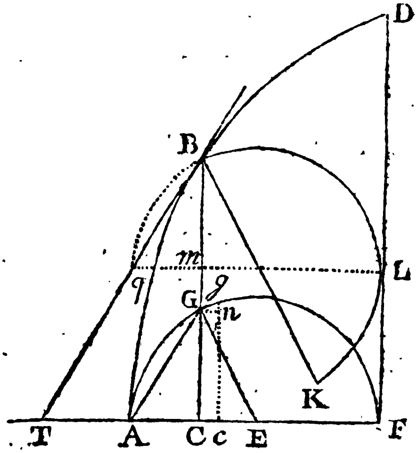
: CA :: AC : CB,) or, becauſe $CE = a - x$, and $\overline{AC \times CE}|^{\frac{1}{2}} = \overline{ax - x^2}|^{\frac{1}{2}} = Cd$, as $\overline{ax - x^2}|^{\frac{1}{2}} : x :: x : y \because x^2 = y \times \overline{ax - x^2}|^{\frac{1}{2}}$ and by ſquaring both Sides of this Equation $x^4 = axy^2 - x^2y^2$, and this divided by x gives $x^3 = ay^2 - xy^2$. In Fluxions it is $3x^2\dot{x} = 2ay\dot{y} - \dot{x}y^2 - 2xy\dot{y}$; then tranſpoſe and divide and it will be $\dot{x} = \frac{2ay\dot{y} - 2xy\dot{y}}{3x^2 + y^2}$ which ſubſtituted for $\dot{x}$ gives the Subtangent CT ($= \frac{\dot{x}y}{\dot{y}}$, *Art.* 35.) $= \frac{2ay^2 - 2xy^2}{3x^2 + y^2} =$ (by ſubſtituting for y^2, its equal $\frac{x^3}{a - x}$) $\frac{2ax - 2x^2}{3a - 2x}$. Whence we may obſerve that AT, the difference between the Abſciſſa AC and Subtangent TC, is $= \frac{ax}{3a - 2x}$.

* This Curve is thus generated. Let A*b*E be a Circle, whoſe Diameter is AE; and make any Arches E*a*, E*b*, and A*c*, A*d*, equal to one another; and thro' the Points *a*, *b*, *d*, *c*, let right Lines be drawn Perpendicular to AE, and tranſverſe Lines from the Point A; then, from A, thro' the Points of Interſection *e*, B, D, &c. draw a Line A*e*BD &c.

and it will be a Curve, called by its Inventor *Diocles*, a Ciſſoid.

EXAMPLE VIII.

45. *To draw a Tangent to the Cycloid* *.



PUT E A the Radius of the generating Circle $=a$, Abſciſs A C $=x$, Ordinate C B $=y$, C G $=s$, and the Arch AG $=z$. Now by the Nature of the Curve C B $=$ C G$+$GA, that is, $y=s+z$. (For, when the generating Circle AGF is in the Poſition B L K, the Arch B L, or GF muſt be equal to L D; and L K, Bq, or GA$=$L F, or m C; but C$m=$G B; therefore GA$=$GB and therefore &c.) and this Equation put into Fluxions gives $\dot{y}=\dot{s}+\dot{z}$.

Let $Gg = z'$, $gn = s'$, and $nG = Cc = \dot{x}$; then, becauſe EG is Perpendicular to Gg, the $\angle$'s gGn and EGC are equal, and therefore the right angled $\triangle$s gnG and GCE are alike; and conſequently $gn : nG :: EC : CG$, that is, $s' : x' :: a-x : s \because s' = \frac{a-x}{s}x'$ or $\dot{s} = \frac{a-x}{s}\dot{x}$, and again, $gG : Gn :: EG : GC$, that is, $z' : x' :: a : s \because z' = \frac{a}{s}x'$ or $\dot{z} = \frac{a}{s}\dot{x}$.

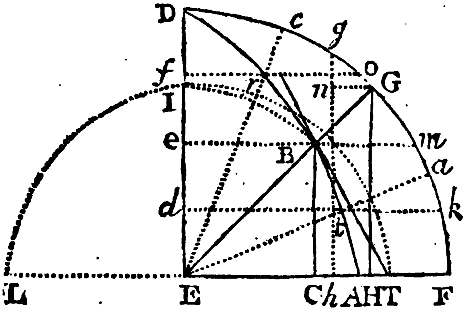
Now by ſubſtituting $\frac{a-x}{s}\dot{x}$ and $\frac{a}{s}\dot{x}$ for $\dot{s}$ and $\dot{z}$ in the above Fluxion of the Equation of the Curve, we have $\dot{y} = \frac{a-x}{s}\dot{x} + \frac{a}{s}\dot{x} = \frac{2a-x}{s}\dot{x}$ therefore the Subtangent CT $(= \frac{\dot{x}y}{\dot{y}}$ *Art.* 35.) $= \frac{sy}{2a-x}$. Whence we may obſerve, by Analogy $2a-x : s :: y : CT$, *i. e.* $FC : CG :: BC : CT$; but by the Property of Circles $FC : CG :: GC : CA \because GC : CA :: BC : CT$; conſequently the $\triangle$s GCA, and BCT are alike, and BT is parallel to GA.

* This Curve is thus generated. Let a Circle, or Wheel, roll along upon a right Line,

until it perform one Revolution, that is, until it meaſure out a right Line equal to its Circumference; then that Point in the Circle which firſt touched the right Line, will deſcribe the Curve called a Cycloid.

EXAMPLE IX.

46. *To draw a Tangent to the Quadratrix of* Dinoſtratus *.



PUT E F, the Radius of the generating Quadrant $= a$; E A the Baſe of the Quadratrix $= b$; Quadrantal Arch F D $= c$; Abſciſs A C $= x$; Ordinate C B $= y$; Arch F G $= z$; and the Sine G H $= s$. Now, by the Nature of the Curve, as Arch F D : Rad. D E :: Arch F G : Ordinate B C (which is evident from the generation of the Curve,) that is, $c : a :: z : y$ $\therefore az = cy$ and $z = \frac{cy}{a}$ in Fluxions $\dot{z} = \frac{c\dot{y}}{a}$. By

Sim. △s BC : CE :: GH : HE, *i. e.* $y : b - x :: s : \frac{sb - sx}{y} = $ HE. Let *gh* be conceived indefinitely near and parallel to GH and G*n* equal and parallel to H*h*, *i. e.* let $ng = s'$ and $gG = z'$, then will the △s EHG and *gn*G be alike, and therefore HE : EG :: *ng* : *g*G, *i. e.* $\frac{sb - sx}{y} : a :: s' : z' = \frac{as'y}{sb - sx}$ or (*Art.* 3.) $\dot{z} = \frac{a\dot{s}y}{sb - sx}$. Hence we have $\frac{a\dot{s}y}{sb - sx} = \frac{c\dot{y}}{a}$ which gives $\dot{s} = \frac{sbc\dot{y} - sxc\dot{y}}{a^2y}$. Now by 47 *E.* 1. $\overline{\overline{EC}^2 + \overline{CB}^2}^{\frac{1}{2}} = $ BE, *i. e.* $\overline{\overline{b - x}^2 + y^2}^{\frac{1}{2}} = $ BE, and by Sim. △s EB : BC :: EG : GH, *i. e.* $\overline{\overline{b - x}^2 + y^2}^{\frac{1}{2}} : y :: a : s = \frac{ay}{\overline{\overline{b - x}^2 + y^2}^{\frac{1}{2}}}$ or $s = \frac{ay}{\overline{b^2 - 2bx + x^2 + y^2}^{\frac{1}{2}}}$ in Fluxions $\dot{s} =$

$$\frac{a\dot{y} \times \overline{\overline{b - x}^2 + y^2}^{\frac{1}{2}} + \frac{b\dot{x}ay - x\dot{x}ay - y\dot{y}ay}{\overline{\overline{b - x}^2 + y^2}^{\frac{1}{2}}}}{\overline{b - x}^2 + y^2} =$$

$\frac{a\dot{y} \times \overline{b - x}^2 + b\dot{x}ay - x\dot{x}ay}{\overline{\overline{b - x}^2 + y^2}^{\frac{3}{2}}} = $ (becauſe $\dot{s}$ by the above is $=$) $\frac{sbc\dot{y} - sxc\dot{y}}{a^2y}$, and this Equation, by

Multiplication and Tranſpoſition, gives $a^3by^2\dot{x} - a^3y^2x\dot{x} = \overline{bcs\dot{y} - csx\dot{y}} \times \overline{\overline{b-x}|^2 + y^2}|^{\frac{1}{2}} - a^3y\dot{y} \times \overline{b-x}|^2$, and this divided by $b-x$ gives $a^3y^2\dot{x} = cs\dot{y} \times \overline{\overline{b-x}|^2 + y^2}|^{\frac{1}{2}} - a^3y\dot{y} \times \overline{b-x}$ (or by ſubſtituting for s its above value $\frac{ay}{\overline{\overline{b-x}|^2 + y^2}|^{\frac{1}{2}}}$, $= acy\dot{y} \times \overline{\overline{b-x}|^2 + y^2} - a^3y\dot{y} \times \overline{b-x}$: But, by a Property of the Curve, as F D : D E :: E F : E A (for when the Arch F G is indefinitely ſmall, it will concide with, or be equal to its Sine G H, and one may be taken for the other; and then E A may be taken for E C, and E F for E H; and therefore then, by the above Nature of the Curve, as Quadrantal Arch FD : Rad. D E :: indefinitely ſmall Sine G H : indef. ſmall Ordinate B C; but by Sim. As G H : BC :: (H E, *i. e.*) E F : (C E, *i. e.*) E A conſeq. Arch F D : Rad. D E :: E F : E A,) *i. e.* as $c : a :: a : b$ ∴ $bc = a^2$, and by ſubſtituting bc for a^2 we have $abcy^2\dot{x} = acy\dot{y} \times \overline{\overline{b-x}|^2 + y^2} - abcy\dot{y} \times \overline{b-x}$ and this Equation divided by acy gives $by\dot{x} = \dot{y} \times \overline{\overline{b-x}|^2 + y^2} - b\dot{y} \times \overline{b-x}$ ∴ $\dot{x} = \dot{y}\,\frac{\overline{b-x}|^2 + y^2 - b \times \overline{b-x}}{by}$ which ſubſtituted for $\dot{x}$ gives the Subtangent C T (=

$\dot{x}y$

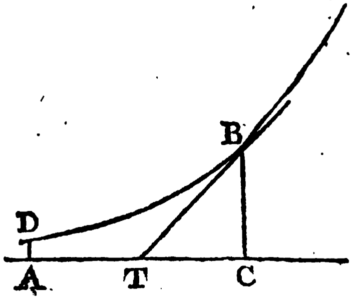
$\frac{\dot{x}y}{\dot{y}}$ *Art.* 35.) $=\frac{\overline{b-x}|^2+y^2}{b}-\overline{b-x}=\frac{\overline{EB}|^2}{EA}-$ EC, which gives the following geometrical Conſtruction, *viz.* if B be the Point to which the Tangent is to be drawn, make E I = E B, and E L = E A, and let the Semicircle L I T be drawn thro' the Points L and I, then T is the Point thro' which the Tangent muſt paſs. For by the Property of Circles LE × ET $=\overline{EI}|^2$ =(by Conſtruction) $\overline{EB}|^2$, *i. e.* EA × $\overline{EC+CT}$ $=EB^2$ ∵ $\frac{\overline{EB}|^2}{EA}=EC+CT$ and $\frac{\overline{EB}|^2}{EA}-EC$ $=CT$.

* This Curve is thus generated. Let EFD be a Quadrant of a Circle, whoſe Radius ED divide into any Number of equal Parts (as E*d*, *de*, *ef*, *f*D) and from the Points of Diviſion (*d*, *e*, *f*,) draw right Lines (*dk*, *e m*, *fo*,) parallel to each other and to the Radius EF: alſo divide the Arch F D into the ſame Number of equal Parts (F *a*, *a* G, G*c* *c* D,) as you do the Radius ED, and to theſe Points of Diviſion (*a*, G, *c*,) draw Radii (E *a*, E G, E *c*,) from the Point or Center E; then a Line drawn from D thro' the Points of Interſection (*r*, B, *t*, &c.)

will be the Curve called a Quadratrix; the Invention of which is imputed to *Dinoſtratus*.

EXAMPLE X.

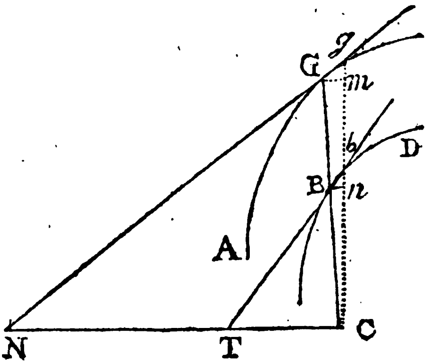
47. *To draw a Tangent to the exponential Curve* DB, *whoſe Equation (putting* AC$=x$, CB$=y$, and $a=$ *a given Quantity) is* $a^x=y$.



PUT A = the hyperbolic Logarithm of a, and Y = the hyperbolic Logarithm of y; then by the Nature of Logarithms $x\mathrm{A}=\mathrm{Y}$, in Fluxions $\dot{x}\mathrm{A}=\dot{\mathrm{Y}}=$ (*ſee Art.* 14.) $\frac{\dot{y}}{y}$ which divided by A gives $\dot{x}=\frac{\dot{y}}{\mathrm{A}y}$ and this ſubſtituted for $\dot{x}$ gives the Subtangent CT $(=\frac{\dot{x}y}{\dot{y}}$ *Art.* 35.) $=\frac{y}{\mathrm{A}y}=\frac{1}{\mathrm{A}}$. Whence we may obſerve that, the Subtangent being an invariable Quantity, the Curve DB is the Logarithmic Curve, (*ſee Art.* 70.)

48. The Examples already given being only of thoſe Curves, which are referred to an Axis, or whoſe Ordinates are parallel to one another; we ſhall now give a few Examples of Spirals, or thoſe whoſe Ordinates all iſſue from one and the ſame Point: Where note the general Expreſſion for the Subtangent will be $\frac{\dot{x}y}{\dot{y}}$ as before, as will be proved in the next Example.

EXAMPLE XI.



49. *Let* AG *be any Curve whoſe Ordinates all iſſue from the fixt Point* C, *and ſuppoſe its Tangent* GN, *Subtangent* NC, *and Ordinate*

CG *to be given; and let there be another Curve, as* B D, *whoſe Ordinates alſo iſſue from the ſame Point* C, *and of ſuch a Nature, that* A G *ſhall always be to* C B *as* a *to* b, *'tis required to draw the Tangent* B T.

N. B. *The Subtangent and Ordinate to all Spirals, are always perpendicular to each other.*

SUPPOSE Cg indefinitely near to C G, that is, let the ∠G Cg be ſuppoſed indefinitely ſmall; and with the Radii CB, CG, let the indefinitely ſmall concentric Arches Bn, Gm, be deſcribed; which being conſidered as coinciding with Tangents to the Points n and m, or as indefinitely ſmall right Lines Perpendicular to Cg, the △s bnB, B C T, will be Similar, as will alſo the △s gmG, G C N, the Curves B D and A G being alſo ſuppoſed to coincide with the Tangents T B and N G in the Points b and g; as they very well may in the very firſt Moment of the Exiſtence of the Angle G Cg, or juſt as it begins to be. Put GN$=c$, NC$=d$, CG$=v$, CB$=y$, AG$=z$, and B$n=x'$; then $nb=y'$, and G$g=z'$. Now bn : nB :: BC : CT, *i. e.* $y' : x' :: y : \frac{x'y}{y'}=$CT, or $\dot{y} : \dot{x}$

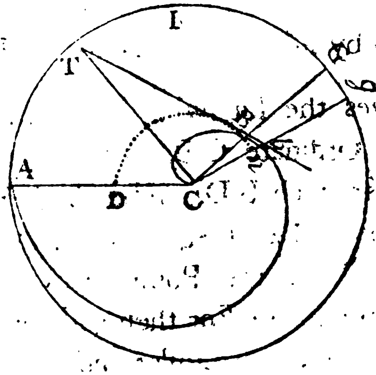
$:: y$

$: : y : \frac{\dot{x}y}{\dot{y}} = CT$, which is a general Expreſſion for the Subtangent of Spirals, or Curves referr'd to a fixt or central Point, and is the ſame as that before found for thoſe Curves which are referred to an Axis, *Art.* 35.

From the ſimilar Sectors or △s CB*n* and CG*m* we have CB : B*n* : : CG : G*m*, *i. e.* $y : x' : : e : \frac{ex'}{y} = Gm$. But by the ſim. △s *gm*G and GCN we have GN : NC : : *g*G : G*m*, *i. e.* $c : d : : z' : \frac{dz'}{c} = Gm$; therefore $\frac{dz'}{c} = \frac{ex'}{y}$ and $z' = \frac{cex'}{dy}$ or $\dot{z} = \frac{ce\dot{x}}{dy}$. Now by the Nature of the Curve AG : CB : : *a* : *b*, that is, $z : y : : a : b$ or $z = \frac{ay}{b}$ in Fluxions $\dot{z} = \frac{a\dot{y}}{b}$. Hence we have $\frac{ce\dot{x}}{dy} = \frac{a\dot{y}}{b}$ and therefore $\dot{x} = \frac{ady\dot{y}}{bce}$ which ſubſtituted for $\dot{x}$ gives the Subtangent CT $(= \frac{\dot{x}y}{\dot{y}}) = \frac{ady^2}{bce}$.

EXAMPLE XII.

50. *To draw a Tangent to the Spiral of* Archimedes *.



PUT the Circumference of the generating Circle $=a$, and its Radius $CA=b$; $CB=y$, Arch $AIG=z$, and $Bn=x'$, $Gg=z'$ (g being ſuppoſed infinitely near to G, and Bn a ſmall Arch concentric to Gg.) Now by the Nature of the Curve $a:b::z:y$ or $bz=ay$, which in Fluxions is $b\dot{z}=a\dot{y}$ or $\dot{z}=\frac{a\dot{y}}{b}$: But by the ſimilar Sectors CBn and CGg, as $CB : Bn :: CG : Gg$, *i. e.* as $y : x' :: b : z'=\frac{bx'}{y}$ or $\dot{z}=\frac{b\dot{x}}{y}$ ∴ $\frac{b\dot{x}}{y}=\frac{a\dot{y}}{b}$ and this Equation reduced duc'd gives $\dot{x} = \frac{ay\dot{y}}{b^2}$ which ſubſtituted for $\dot{x}$ gives the Subtangent CT ($= \frac{\dot{x}y}{\dot{y}}$, *Art.* 48.) $= \frac{ay^2}{b^2} = \frac{yz}{b}$ by ſubſtituting bz for its Value ay; which gives the following Conſtruction, *viz.* with the Ordinate C B, as a Radius deſcribe the circular Arch B D, and make C T perpendicular to C G and equal to the Arch B D; then will T be the Point thro' which the Tangent muſt paſs: For then the Sectors CGA and C B D will be Similar and conſeq. C G : G A :: C B : B D, *i. e.* $b : z :: y :$ B D $= \frac{yz}{b}$ = C T.

N. B. The above Expreſſion for the Subtangent may be inferred from that in *Art.* 49, for if in that Article, the Curve A G be an Arch of a Circle, the Tangent G N and Subtangent N C will both be infinite (ſince the Tangent to a Circle is perpendicular to the Radius, and conſeq. C G being the Radius, G N and N C will be parallel;) and a denote the Circumference, and b the Radius of that Circle; and therefore b and e will be equal; and ſubſtituting b for e and ſtriking c and d out of $\frac{ady^2}{bce}$, the Expreſſion for the Subtangent; we ſhall have $\frac{ay^2}{b^2}$ = the Subtangent as above.

* This Curve is thus generated. With the Radius C A, let the Circle A I G A be deſcribed with an equable Motion, or, the Point A deſcribe equal Arches in equal Times; and at the ſame Moment of Time that the Point A begins to generate the Circle, let another Point be conceived to begin to move along the Radius C A from C towards A, and to paſs over it with an uniform Motion, and ſuch Velocity that it may arrive at A at the very ſame Moment of Time that the Radius C A ſhall have deſcribed the Circle, or come to be in its firſt Situation: Then will the Point moving along the Radius C A, generate, or deſcribe, the Curve C B *b* A called a Spiral, which Name was given to it by its Inventor, *Archimedes*.

CHAP.

CHAP. V.

Of finding the Points of Inflection or contrary Flexure in CURVES.

HITHERTO we have had no Occaſion to make uſe of ſecond Fluxions, and therefore have not treated of them: But here we ſhall ſay ſomething concerning them, as they will be uſed in this, and the following Chapter.

51. IN all fluxional Equations, it is proper that ſome one of the variable Quantities be ſuppoſed to encreaſe uniformly, with which the others may be always compared; which Quantity will therefore have no ſecond Fluxion, that is, the firſt Fluxion of it will be invariable. Now the Reaſon of this is plain; for if we only know in general, that two or more Bodies, are carried over unequal Spaces in unequal Times, no Inference can from thence be drawn, with regard to the Equality or Inequality of their Velocities; but if the Spaces paſs'd over, are compared with the Times in which the Bodies were moving, we ſhall then obtain the Ratio of the Velocities of the moving Bodies: Now in Order to this Compariſon, there muſt neceſſarily be ſome Standard which is the

common Meaſure of the Times the Bodies were in Motion ; the ſame as, in the Compariſon of two or more Magnitudes, there muſt be always ſome Quantity, which is the common Meaſure of all. And, in Curves, when we make the Fluxion of any flowing Quantity, as the Abſciſs for Inſtance, invariable, it is, in Effect, no more than taking equal Portions of it, that we may thereby determine the Fluxions of the other variable Quantities, or the Proportions with which they flow.

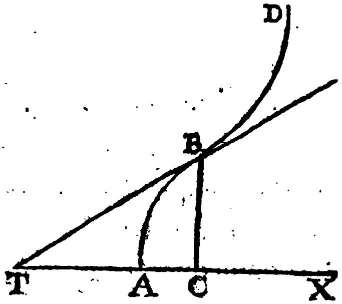
52. In finding the ſecond Fluxion of any Equation, all the variable Quantities and Fluxions, muſt be conſidered as diſtinct Fluents, and fluxed accordingly, by the Rules laid down in *Chap.* 2. Thus the Fluxion of the Equation $z\dot{y}=a\dot{x}$, by *Art.* 9. will be $\dot{z}\dot{y}+z\ddot{y}=a\ddot{x}$, but as we my aſſume either $\dot{x}$ or $\dot{y}$ as invariable, by making $\dot{x}$ ſo, it will be $\dot{z}\dot{y}+z\ddot{y}=0$, becauſe $\ddot{x}$, and conſequently $a\ddot{x}$, is $=0$, or if we make $\dot{y}$ invariable, it will be $\dot{z}\dot{y}=a\ddot{x}$, becauſe then $\ddot{y}$ and conſequently $z\ddot{y}$ will be $=0$. Again the Fluxion of $\frac{a\dot{z}x}{\dot{x}^2}+\frac{y\dot{y}}{a}=\dot{y}$, if we make $\dot{y}$ invariable or $\ddot{y}=0$, will be $\frac{a\ddot{z}x\dot{x}^2+a\dot{z}\dot{x}^3-2\dot{x}\ddot{x}a\dot{z}x}{\dot{x}^4}+\frac{\dot{y}^2}{a}=\ddot{y}=0$, that is,

$a\ddot{z}x\dot{x}+$

$\frac{a\ddot{z}x\dot{x}+a\dot{z}\dot{x}^2-2\ddot{x}a\dot{z}x}{\dot{x}^3}+\frac{\dot{y}^2}{a}=0$. The third or fourth Fluxions, &c. are found in the ſame Manner, due regard being had to ſuch Fluxions as are ſuppoſed invariable.

53. When a Curve from being Concave becomes Convex towards its Axis, or from being Convex becomes Concave, that Point where the Change is made, or that which ſeparates the Convex from the Concave Part, is called the Point of Inflection, or contrary Flexure; or, the Point of Inflection is that, to which a Tangent being drawn, cuts the Curve.

54. In the Curve A B D, ſupppoſe B to be the Point of Inflection; then, becauſe A B is



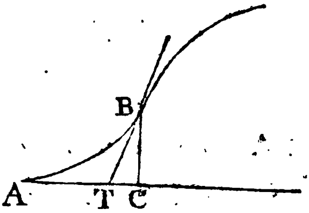
Concave and B D Convex towards the Axis A X, is it evident, that the indefinitely ſmall Increment of the Curve at the ſaid Point B, will be neither Concave nor Convex, that is, will be a right Line; and becauſe, during this Increment

crement the Tangent at B will exactly concide with the Curve, or the Ordinate will flow on each Side of the Point B with an equable or uniform Motion, the Abſciſs being ſuppoſed to flow ſo too, it follows, that the Ordinate will by flowing along that Increment, have no variation of Increaſe, or its ſecond Fluxion will be $=0$. Or, becauſe when the Abſciſs flows on with an uniform Motion, the Ordinate flows with a retarded Motion, when the Curve is Concave towards the Abſciſs, and with an accelerated Motion when it is Convex, therefore at the Point of Inflection, where the Curve is neither Concave nor Convex the Ordinate muſt flow, with neither a retarded nor accelerated, but, with an uniform Motion, and conſequently its ſecond Fluxion muſt be equal to Nothing. Therefore,

55. PUT the Equation of the given Curve (where $AC=x$ and $CB=y$) into Fluxions, and find the Value of $\dot{x}$ or $\dot{y}$; and put this Value of $\dot{x}$ or $\dot{y}$ into Fluxions again, and make both $\ddot{x}$ and $\ddot{y}=0$; expunge the reſt of the fluxional Quantities, by the help of the Fluxion of the Equation of the Curve; and you will have the Point of Inflection ſought, determined.

56. *Note*, When the Curve is firſt Concave

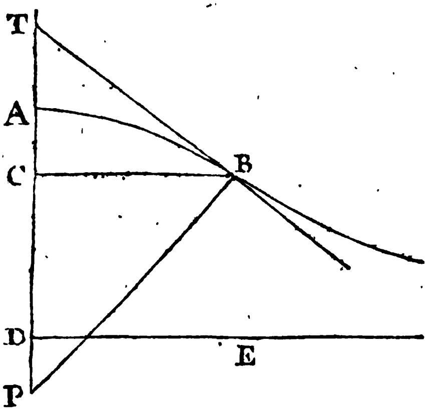
towards the Axis, as in the laſt Figure, it is evident, if B be the Point of Inflection, and BT a Tangent to it, that AT the Difference between the Subtangent and Abſciſs will be a *Maximum:* and when the Curve is firſt Convex towards the Axis, as in the annexed Figure, it is evident, that AT the Difference be-



tween the Abſciſs and Subtangent will be alſo a *Maximum.* Wherefore the Point of Inflection may be determined without the help of ſecond Fluxions, by firſt finding a definitive Expreſſion for the Subtangent, by Chap. 4. and the Difference between that and the Abſciſs, and then making the Fluxion of this Difference equal to Nothing. This we ſhall illuſtrate in the following Example.

EXAMPLE I.

57. *To find the Point of Inflection in the Conchoid of* Nicomedes, *or, the Point where it begins to be Convex towards the Aſſymptote* DE.



N. B. At A the Curve is Concave towards DE; but it cannot poſſibly continue ſo long; for, if it did, the farther it proceeded the more it would incline or tend towards the ſaid Line, and ſo interſect it; whereas in fact, the farther it proceeds, the leſs it tends towards the ſaid Line: Conſequently the Curve muſt have a Point of Inflection.

PUT $PD=a$, $DA=b$, $DC=x$, and $CB=y$; then by *Art.* 43. the Fluxion of the Equation of the Curve will be $\dot{y}=\frac{-ab^2\dot{x}-x^3\dot{x}}{x^2\cdot\times\overline{b^2-x^2}|^{\frac{1}{2}}}$ Suppoſe B to be the Point of Inflection, then the Fluxion of this again, (becauſe x being ſuppoſed to flow with an equable Motion, the ſecond Fluxions of x and $y=0$ will be $0=$

$-3x^4$

$$\frac{-3x^4\dot{x}^2\times\overline{b^2-x^2}|^{\frac{1}{2}}-2x\dot{x}\times\overline{b^2-x^2}|^{\frac{1}{2}}-\frac{x^3\dot{x}}{\overline{b^2-x^2}|^{\frac{1}{2}}}}{x^4}$$

$$\frac{\overline{-ab^2\dot{x}-x^3\dot{x}}}{\times\overline{b^2-x^2}},\ i.\ e.\ 0=\frac{2ab^4x-3ab^2x^3-b^2x^4}{x^4\times\overline{b^2-x^2}|^{\frac{1}{2}}}\dot{x}^2$$

which multiplied by $x^4\overline{b^2-x^2}|^{\frac{1}{2}}$ and divided by $b^2x\dot{x}^2$ gives $0=2ab^2-3ax^2-x^3$, or $x^3+3ax^2-2ab^2=0$, by which x and conſequently the Point B may be determined: And if $a=b$ it will be $x^3+3ax^2-2a^3=0$, which divided by $x+a$ gives $x^2+2ax-2a^2=0$, or $x^2+2ax=2a^2$, and by ſolving the Quadratic $x=\overline{3a^2}|^{\frac{1}{2}}-a$.

Or, becauſe the Subtangent CT (*Art.* 43.) is $=\frac{\overline{a+x}\times\overline{b^2x-x}}{ab^2+x^3}$, and AC is $=b-x$; therefore AT (=CT—CA) $=\frac{\overline{a+x}\times\overline{b^2x-x^3}}{ab^2+x^3}-b+x=$ *a Maximum* (*Art.* 56.) which thrown into Fluxions is

$$\frac{\overline{ab^2\dot{x}-3ax^2\dot{x}+2b^2x\dot{x}-4x^3\dot{x}}\times\overline{ab^2+x^3}}{\overline{ab^2+x^3}|^2}\text{ or}$$

$$\frac{-:3x^2\dot{x}\times\overline{ab^2x-ax^3+b^2x^2-x^4}}{a^2b^4+2ab^2x^3+x^6}+\dot{x}=0,$$

then multiply by $a^2b^4+2ab^2x^3+x^6$ and ſtrike out the contradictory Terms, and it will be $2a^2b^4\dot{x}-4ab^2x^3\dot{x}-3a^2b^2x^2\dot{x}+2ab^4x\dot{x}-b^2x^4\dot{x}$

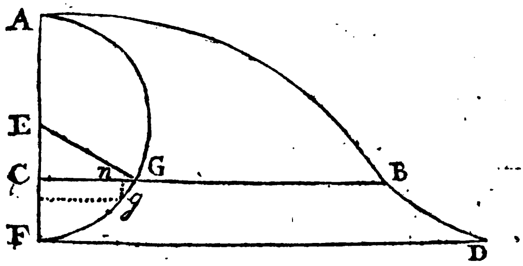
$=0,$

$=0$, which divided by $-ab^2\dot{x}-b^2x\dot{x}$ gives $x^3+3ax^2-2ab^2=0$, as before.

We may obſerve here the ſtrict Agreement in theſe two Methods of Solution; ſince, tho' the Premiſſes are widely different, yet the Concluſions are exactly the ſame: And indeed, in all fluxional Operations, where the Thing ſought may be obtained by different Methods of reaſoning, theſe Methods do never ſerve to give us different Expreſſions of one and the ſame Value, but always the very ſame Expreſſion; unleſs in ſome Caſes where there is any Ambiguity in the algebraic Operation.

EXAMPLE II.

58. *To find the Point of Inflection in the inflected Cycloid *, the Circumference of whoſe generating Circle is to its Baſe as 4 to 5.*



PUT the generating Semicircle $AGF=a$, Baſe $FD=b$, Radius EA or $EF=r$, Abſciſs

$AC=x$, Ordinate $CB=y$, Arch $AG=z$, and Sine $GC=s$; and ſuppoſe B to be the Point of Inflection ſought. Now, by the Nature of the Curve $a:b::z:(GB)\ y-s$ ∵ $ay-as=bz$, and $y=s+\frac{bz}{a}$ and the Fluxion of this Equation is $\dot{y}=\dot{s}+\frac{b\dot{z}}{a}$. Let the fluxional Triangle Ggn and the Radius EG be drawn; then will the Triangles ECG and Gng be alike; (for Gg being conſidered as an indefinitely ſmall right Line coinciding with the Tangent to the Point G, the $\angle EGg$ will be a right Angle, and therefore $=\angle CEG+\angle EGC$, and the $\angle EGC$ being common, therefore $\angle CEG=\angle nGg$ and conſeq. the $\angle sGCE$ and Gng being right, the $\angle EGC=$ the $\angle Ggn$:) therefore, as $nG:ng::EC:CG$, or, $\dot{x}$ and $\dot{s}$ being Negative to each other, as $-\dot{s}:\dot{x}::x-r:s$ ∵ $\dot{s}=\frac{\overline{r-x}\times\dot{x}}{s}$, but by the Property of the Circle $GC=\overline{AC\times CF}\big|^{\frac{1}{2}}$, *i. e.* $s=\overline{2rx-x^2}\big|^{\frac{1}{2}}$ which ſubſtituted for s gives $\dot{s}=\frac{\overline{r-x}\times\dot{x}}{\overline{2rx-x^2}\big|^{\frac{1}{2}}}$. By 47 *E.* 1. $\dot{z}=\overline{\dot{x}^2+\dot{s}^2}\big|^{\frac{1}{2}}$ (*i. e.* by ſubſtituting for $\dot{s}$ its Value) $=\frac{r\dot{x}}{\overline{2rx-x^2}\big|^{\frac{1}{2}}}$.

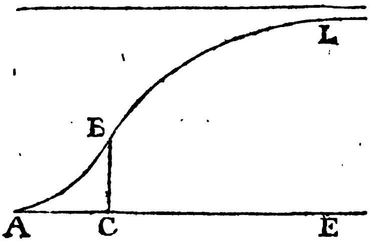
Whence $\dot{y}$ $\left(=\dot{s}+\frac{b\dot{z}}{a}\right)=\frac{\overline{r-x}\times\dot{x}}{\overline{2rx-x^2}|^{\frac{1}{2}}}+\frac{br\dot{x}}{a\times\overline{2rx-x^2}|^{\frac{1}{2}}}=\frac{ar+br-ax}{a\times\overline{2rx-x^2}|^{\frac{1}{2}}}\dot{x}$. And this put into Fluxions, ſuppoſing x to flow with an uniform Motion, or $\ddot{x}$ and $\ddot{y}=0$, (*Art.* 54.) is $0=\frac{brx-ar^2-br^2}{a\times\overline{2ax-x^2}|^{\frac{3}{2}}}\dot{x}^2$, conſequently $brx-ar^2-br^2=0$, and $brx=ar^2+br^2 \because x=\frac{a+b}{b}r$, and $EC\ (=x-r)=\frac{a}{b}r$, which gives the Point C; from which a perpendicular Ordinate being drawn, will fall on the Point of Inflection: And this is an univerſal Theorem for all inflected Cycloids, when $\frac{a}{b}$ expreſſes the Ratio of the Circumference of the generating Circle to the Baſe of the Cycloid.

* See how this Curve may be generated, *Chap.* 14. *Queſt.* 7.

EXAMPLE III.

59. *To find the Point of Inflection* B *in the Curve* A B D, *whoſe Equation (putting* $AC=x$, $CB=y$, *and* $a=$ *a given Quantity) is* $ax^2=a^2y+x^2y$.

THE



THE Equation of the Curve put into Fluxions is $2ax\dot{x}=a^2\dot{y}+2x\dot{x}y+x^2\dot{y}$, therefore $\dot{y}=\frac{2ax\dot{x}-2x\dot{x}y}{a^2+x^2}$ (*i. e.* by ſubſtituting $\frac{ax^2}{a^2+x^2}$ for its Value y) $=\frac{2a^3x\dot{x}}{\overline{a^2+x^2}|^2}$, and the Fluxion of this again, making $\ddot{x}$ and $\ddot{y}=0$, is $0=\frac{2a^3\dot{x}^2\times\overline{a^2+x^2}|^2-\overline{4a^2x\dot{x}+4x^3\dot{x}}\times 2a^3x\dot{x}}{\overline{a^2+x^2}|^4}$, theref. $2a^3\dot{x}^2\times\overline{a^2+x^2}|^2 - : \overline{4a^2x\dot{x}+4x^3\dot{x}}\times 2a^3x\dot{x}=0$ and by dividing by $2a^3\dot{x}$, we ſhall have $\dot{x}\times\overline{a^2+x^2}|^2-4a^2x^2\dot{x}-4x^4\dot{x}=0$, therefore $4x^4\dot{x}+4a^2x^2\dot{x}=\dot{x}\times\overline{a^2+x^2}|^2$, and this Equation divided by $x^2\dot{x}+a^2\dot{x}$ gives $4x^2=a^2+x^2$ by tranſpoſition $3x^2=a^2$ ∵ $x^2=\frac{1}{3}a^2$ and $x=a\sqrt{\frac{1}{3}}$ and if this be ſubſtituted for x in the Equation of the Curve, we ſhall have y or the Ordinate at the Point of Inflection $=\frac{1}{4}a$.

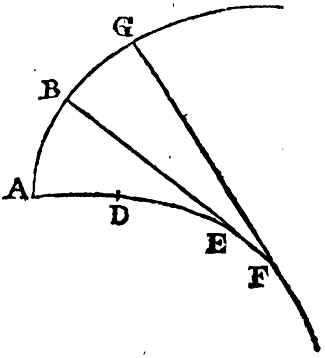
If it were required to find the Aſſymptote of this Curve, we need only ſuppoſe the Abſciſs and Curve to be indefinitely extended; and then becauſe x^2 will be indefinitely near to Equality with a^2+x^2, we ſhall have y, which by the Equation of the Curve is $=a\times\frac{x^2}{a^2+x^2}$, indefinitely near to Equality with the given right Line a, that is y, will then be $=a$ leſs a Quantity indefinitely ſmall: Wherefore if a right Line, to the Diſtance of the given right Line a, be drawn parallel to the Diameter AE it will be the Aſſymptote required.

CHAP. VI.

Of finding the Radius of Curvature in Curves.

60. AS the Curvature or Convexity of all Curves, but Circles, varies in every Point; therefore, if Circles are drawn to concide with the given Curve in any Number of Points, or, which is the ſame, ſo that the Tangents to the Circles may concide with the Tangents to the Curve, the Radii of theſe Circles will be different: And the finding of theſe Radii is the Buſineſs of this Chapter.

And becauſe all Curves are formed, or may be formed, or generated, by the Evolution or winding off of ſome other Curves; therefore the Centers of theſe Circles which coincide, and conſequently have an equal Curvature with the different Points, or rather Increments, of theſe Curves, will be continually in the Curves to be unwound; which Curves are called the Evolutes, and the others formed or generated, or conceived to be generated, by their Evolution, are called the Involutes.



61. To make this plainer: Let DEF be any Curve, round which conceive a Thread to be wound and extended beyond the Curve D in a right Line to A: Let this Thread be evolved, or wound off, from the Curve DE, ſo that it be continually ſtretched at its full Length as it leaves the Curve; then will the Point A generate, or deſcribe the involute Curve ABG: and AD, BE, GF, will be the Radii of Curvature at the Points A, B, G,

reſpectively,

reſpectively. From this Definition, we may draw theſe following

COROLLARIES.

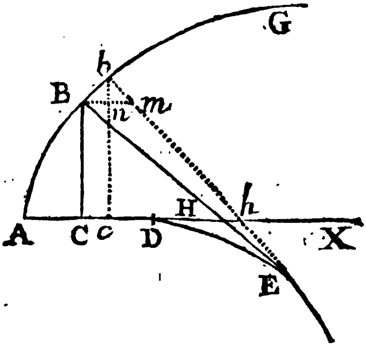
1. The Radius of Evolution or Curvature BE, will be always equal to the Length of the Curve DE, and the right Line AD; and conſequently, if the vertical Diſtance or ſhorteſt Radius AD vaniſh, *i. e.* if the Radius at A be nothing; then the involute Curve will begin at D and ſo the Curve DE will be equal to the Radius of Evolution or Curvature at the Point B.

2. Becauſe the Radius of a Circle is perpendicular to the Tangent, the Radius of Curvature at any Point B is always perpendicular to a Tangent to the Curve at that Point.

3. The ſame Radius BE, which is perpendicular to the Involute at the Point B, is alſo a Tangent to the Evolute at the Point E.

62. *To deduce a general Expreſſion for* BE, *the Radius of Evolution or Curvature, for any Point* B *of the involute Curve* ABG, *whoſe Axis is* AX, *and Evolute* DE.

PUT the Abſciſs $AC=x$, Ordinate $CB=y$; ſuppoſe bE indefinitely near to BE, bc indefinitely near and parallel to BC, and Bm parallel to AX, *i. e.* let Cc or $Bn=x'$, and $nb=y'$;



$nb=y'$; then by 47 *E.* 1. $Bb=\overline{x'^2+y'^2}\Big|^{\frac{1}{2}}$ Now becauſe EB is perpendicular to a Tangent at the Point B, and B*b* is ſuppoſed to coincide with that Tangent, the Triangles B*nb* and BCH are Similar (For ∠EB*b*= CB*n*, and therefore, the ∠EB*n* being common, the Angles *n*B*b* and CBH are equal ∵ ∠B*bn*= ∠BCH, *ergo*, &c.) and the ∠B*bm* being right, the Triangles *mnb* and *bn*B are ſimilar alſo, as are likewiſe therefore the Triangles *bnm* and BCH. Wherefore B*n* : *nb* :: BC : CH, *i. e.* $x' : y' :: y : \frac{yy'}{x'} = CH$: And therefore (by 47 *E.* 1.) BH= $\overline{\overline{BC}\big|^2+\overline{CH}\big|^2}\Big|^{\frac{1}{2}}=\overline{y^2+\frac{y^2y'^2}{x'^2}}\Big|^{\frac{1}{2}}=\frac{y}{x'}\times\overline{x'^2+y'^2}\Big|^{\frac{1}{2}}$ and $AH=x+\frac{yy'}{x'}$. Again B*n* : *nb* :: *bn* : *nm* *i. e.* $x' : y' :: y' : \frac{y'^2}{x'} = nm \because Bm = x' + \frac{y'^2}{x'}$. Now becauſe the Direction of the Curve ABG approaches continually nearer to a Paralleliſm with the Axis AX; if we ſuppoſe the Abſciſs (x) to flow with an equable or uniform Motion; that is, ſuppoſing x' or $\dot{x}$ to be invariable; it is plain that the Increment of the Ordinate (y) or the Velocity, with which it flows, muſt continually decreaſe; and therefore the Increment or Fluxion of this Increment, or the ſecond Increment of y, will be Negative. Therefore Hh, the Increment of AH, *viz.* the Increment of $x + \frac{yy'}{x'}$ will be $x' + \frac{y'^2 - yy''}{x'}$. Now the Triangles EB$m$ and and EHm are ſimilar, as is evident; therefore Bm—Hh : Bm :: (BE—HE=) BH : BE, *i. e.* $\frac{yy''}{x'} : x' + \frac{y'^2}{x'} :: \frac{y}{x'} \times \overline{x'^2 + y'^2}|^{\frac{1}{2}} : \frac{\overline{x'^2 + y'^2}|^{\frac{3}{2}}}{x'y''}$ = BE : or, ſubſtituting the Fluxion for the Increment (*Art.* 3.) BE $= \frac{\overline{\dot{x}^2 + \dot{y}^2}|^{\frac{3}{2}}}{\dot{x}\ddot{y}}$.

63. *Note,* If x flow with an uniform and y with an eccelerated Motion, *i. e.* if $\dot{x}$ be invariable,

riable, as above, and the Fluxion of $\dot{y}$ affirmative (as it will be when the Curve is Convex towards its Axis) the general Expreſſion for BE, the Radius of Curvature, will be $\frac{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$.

64. Hence, becauſe we may ſubſtitute 1 for any invariable Fluxion, if we put $\dot{x}=1$, the general Expreſſion for BE the Radius of Curvature will be $=\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{\ddot{y}}$ when the Fluxion of $\dot{y}$ is Negative, or the Curve is Concave towards its Axis; and $=\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{-\ddot{y}}$ when the Fluxion of $\dot{y}$ is Affirmative, or the Curve is Convex towards its Axis. Wherefore, if we put the Equation of the given Curve, which expreſſes the Relation between the Abſciſs and Ordinate into Fluxions, making $\dot{x}=1$; and put this fluxional Equation into Fluxions again, ſtill ſubſtituting 1 for $\dot{x}$, and making the Fluxion of $\dot{y}$ Negative when the Curve is Concave, and Affirmative when Convex towards its Axis, the Values of the *Second* and *Square* of the *Firſt* Fluxion of y may be had or determined; which therefore being ſubſtituted for them in one of theſe general Expreſſions, *viz.* in the firſt, when the Fluxion of $\dot{y}$ is Negative, and in

the ſecond, when Affirmative, will give a definitive Expreſſion for B E, or the Radius of Curvature required.

65. *N. B.* The ſhorteſt Radius, or vertical Diſtance A D may be obtained by ſubſtituting the Values of $\dot{x}$ and $\dot{y}$ in the general Expreſſion for the ſubnormal CH, which was found (*Art.* 62.) $=\frac{yy'}{x'}$ or (ſubſtituting $\dot{x}$ for x' and $\dot{y}$ for y') $\frac{y\dot{y}}{\dot{x}}$; and then making x and y vaniſh in the definitive Expreſſion which will be then found: For the Expreſſion for C H being the ſame at whatever Point of the Curve B is taken, therefore if it be taken at A where x and y vaniſh or become $=0$, the Point C muſt of conſequence coincide with the Point A, and the Points E and H with D, *ergo*, &c.

66. *N. B.* The ſubſtituting Unity, or 1, rather than any other Number, for an invariable Fluxion, has no manner of Effect on the Working, otherwiſe than by making the Operation much leſs laborious; and, in reality, it is no more than making Unity the Standard of the other Fluxions, or reducing the other Fluxions to a Compariſon with 1.

Ex-

EXAMPLE I.

67. *To find a definitive Expreſſion for the Radius of Curvature* BE *(ſee the laſt Fig.) as alſo* AD *the Diſtance of the Vertices of the Evolute and Involute Curves; ſuppoſing the involute Curve* ABG *to be the common or apollonian Parabola.*

THE Equation of the common Parabola is $ax=y^2$ (*ſee Art.* 38.) which in Fluxions is $a\dot{x}=2y\dot{y}$; or, if we write 1 for $\dot{x}$, it is $a=2y\dot{y}$; therefore $\dot{y}=\frac{a}{2y}=\frac{a}{2\times\overline{ax}\big|^{\frac{1}{2}}}$, for y is $=ax^{\frac{1}{2}}$ by the Equation of the Curve: Now the Fluxion of this Equation again (the Curve being Concave towards its Axis, and therefore the Fluxion of $\dot{y}$ Negative) is $-\ddot{y}=\frac{-a^2}{4\times\overline{ax}\big|^{3}}$ whence $\dot{y}^2=\frac{a^2}{4\times ax}=\frac{a}{4x}$ and $\ddot{y}=\frac{a^2}{4\times\overline{ax}\big|^{\frac{3}{2}}}$. Now, if for $\dot{y}^2$ and $\ddot{y}$, we ſubſtituted theſe their Values, we ſhall have $\frac{\overline{1+\dot{y}^2}\big|^{\frac{3}{2}}}{\ddot{y}}$, the general Expreſſion for the Radius of Curvature BE (*Art.* 64.) $=$

1+

$\frac{\overline{1+\frac{a^2}{4ax}}|^{\frac{3}{2}} \times 4 \times \overline{ax}|^{\frac{1}{2}}}{a^2} = \frac{\overline{4ax+a^2}|^{\frac{3}{2}}}{2a^2}$, which is the definitive Expreſſion required. And by ſubſtituting $\frac{a}{2y}$ for $\dot{y}$ in ($\frac{y\dot{y}}{\dot{x}}$ *Art.* 65. or $\dot{x}$ being 1.) $y\dot{y}$ we have $\frac{a}{2}$ or $\frac{1}{2}a=$A D the vertical Diſtance; which ſame truth may be infer'd from the Expreſſion for B E, for when the Radius becomes the vertical Diſtance, that is, when the Point B coincides with A, x vaniſhes, and therefore ſtriking $4ax$ out of the ſaid Expreſſion, we have $\frac{\overline{a^2}|^{\frac{3}{2}}}{2a^2}=\frac{1}{2}a=$A D as before.

EXAMPLE II.

68. *Let* $y^m=x$ *expreſs the Nature of all Parabolas univerſally* (ſee Art. 39.) *to find the Radius of Evolution or Curvature, and vertical Diſtance; as alſo the Conſequences on theſe three Suppoſitions reſpectively,* viz. $m=2$, *m more than* 2, *m leſs than* 2.

THE Fluxion of this Equation, making $\dot{x}=1$, is $my^{m-1}\dot{y}=1$; and the Fluxion of this again (ſuppoſing $\dot{y}$ Negative) is $\overline{m-1}\times my^{m-2}\dot{y}^2$

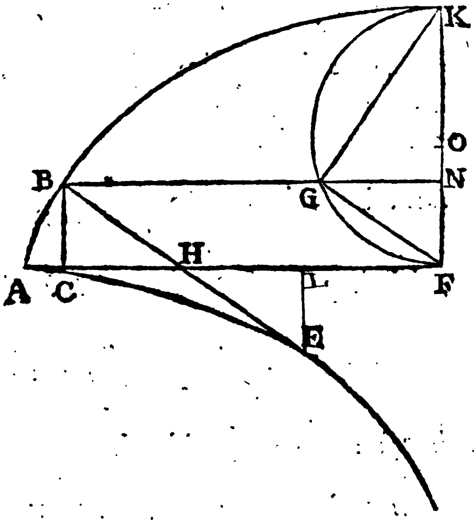
$-my^{m-1}\dot{y}=0$, therefore $\dot{y}=\frac{1}{my^{m-1}}$, and $\dot{y}^2=\frac{1}{m^2y^{2m-2}}$, and $\ddot{y}=\frac{\overline{m-1}\times my^{m-2}}{my^{m-1}}\dot{y}^2=$ (by dividing both Numerator and Denominator by my^{m-2}) $\frac{m-1}{y}\dot{y}^2=$ (by writing for $\dot{y}^2$ its equal) $\frac{m-1}{m^2y^{2m-1}}$. Now by ſubſtituting for $\dot{y}^2$ and $\ddot{y}$ theſe their Values, we ſhall have the general Expreſſion for the Radius of Curvature $\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{\ddot{y}}$

(*Art.* 64.) $=\frac{\overline{1+\frac{1}{m^2y^{2m-2}}}|^{\frac{3}{2}}\times m^2y^{2m-2}}{m-1}=$

$\frac{\overline{m^2y^{2m-2}+1}|^{\frac{3}{2}}}{\overline{m-1}\times\frac{1}{m}y^{2-m}}=$ the Radius of Curvature required. And by ſubſtituting $\frac{1}{my^{m-1}}$ for $\dot{y}$ in ($\frac{y\dot{y}}{\dot{x}}$, or, $\dot{x}$ being $=1$,) $y\dot{y}$ (*ſee Art.* 65.) we have the Subnormal $=\frac{y}{my^{m-1}}=\frac{y^2}{my^m}=\frac{y^{2-m}}{m}$, *i. e.* (if $m=2$,) $=\frac{y^0}{2}=\frac{1}{2}$; if m be more than 2, then $2-m$ will be a negative Index, and conſequently the proper Place of y will be in the Denominator, and therefore if y be made $=0$, it will be infinite and equal to the vertical Diſtance: And laſtly, if m be leſs than 2, then $2-m$ is a poſitive Index, and y's proper Place is in the Numerator, as above, and therefore when the Subnormal is = the vertical Diſtance, or, $y=0$, the ſaid vertical Diſtance will be nothing.

EXAMPLE III.

69. *To find the Radius of Curvature in that moſt beautiful mechanical Curve the Cycloid; as alſo the Diſtance of its Vertex from that of the Evolute.*



PUT OF or OK$=a$, AC$=x$, CB$=$FN y, Sine NG$=s$, Arch GF$=z$. Now by the Nature of the Curve (*ſee Art.* 45.) the Arch KG $=$ GB and therefore the Arch FG$=$GN$+$AC, or AC$=$ Arch FG$-$GN, *i. e.* $x=z-s$, or, (becauſe by the Property of Circles NG $=\overline{\mathrm{KN}\times\mathrm{NF}}|^{\frac{1}{2}}$, *i. e.* $s=\overline{2ay-y^2}|^{\frac{1}{2}}$,) $x=z-\overline{2ay-y^2}|^{\frac{1}{2}}$ which in Fluxions, making $\dot{x}=1$, is $1=\dot{z}+\frac{y\dot{y}-a\dot{y}}{\overline{2ay-y^2}|^{\frac{1}{2}}}$, but $\dot{z}=\overline{\dot{s}^2+\dot{y}^2}|^{\frac{1}{2}}=$ $\overline{\frac{\overline{a\dot{y}-y\dot{y}}|^2}{2ay-y^2}+\dot{y}^2}|^{\frac{1}{2}}=\frac{a\dot{y}}{\overline{2ay-y^2}|^{\frac{1}{2}}}$, therefore $1=$ $\frac{a\dot{y}}{\overline{2ay-y^2}|^{\frac{1}{2}}}+\frac{y\dot{y}-a\dot{y}}{\overline{2ay-y^2}|^{\frac{1}{2}}}$, *i.e.* $1=\frac{y\dot{y}}{\overline{2ay-y^2}|^{\frac{1}{2}}}\therefore\dot{y}=$ $\frac{\overline{2ay-y^2}|^{\frac{1}{2}}}{y}$ and this in Fluxions again, becauſe the Fluxion of $\dot{y}$ is Negative, is $-\ddot{y}=$

$$\frac{\frac{ay\dot{y}-y^2\dot{y}}{\overline{2ay-y^2}|^{\frac{1}{2}}}-\dot{y}\times\overline{2ay-y^2}|^{\frac{1}{2}}}{y^2}=\frac{-a\dot{y}}{y\times\overline{2ay-y^2}|^{\frac{1}{2}}}=$$

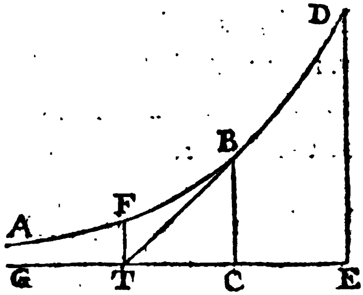
(by ſubſtituting for $\dot{y}$, its equal,) $-\frac{a}{y^2}$, or $\ddot{y}=$ $\frac{a}{y^2}$. Now by ſubſtituting $\frac{2ay-y^2}{y^2}$ for $\dot{y}^2$ and $\frac{a}{y^2}$ for $\ddot{y}$ we ſhall have, by *Art.* 46. $\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{\ddot{y}}=$

$\frac{\overline{1+\frac{2ay-y^2}{y^2}}|^{\frac{3}{2}}\times y^{2}}{a}=\frac{\overline{2ay}|^{\frac{3}{2}}}{ay}=$ B E the Radius of Curvature required: Whence, by Analogy B E : $\overline{2ay}|^{\frac{3}{2}}$ :: $2ay$: ay, that is, B E : $\overline{2ay}|^{\frac{1}{2}}$ :: 2 : 1, but $\overline{2ay}|^{\frac{1}{2}}=$ F G (for by 47 *E.* 1. $\overline{GN}|^2+\overline{NF}|^2=\overline{FG}|^2$, and by the Property of Circles $\overline{GN}|^2=KN\times NF$, therefore $\overline{FG}|^2=KN\times NF+\overline{NF}|^2=\overline{2a-y}\times y+y^2=2ay$ and $FG=\overline{2ay}|^{\frac{1}{2}}$,) therefore B E = 2G F: And as a Tangent at the Point B is parallel to the Chord K G, by *Art.* 45. and the Angle KGF is right, therefore, by *Art.* 61. *Cor.* 2. B E will be parallel to the Chord G F.—By *Art.* 65. by ſubſtituting $\frac{\overline{2ay-y^2}|^{\frac{1}{2}}}{y}$ for its equal $\dot{y}$ in ($\frac{y\dot{y}}{\dot{x}}$, or, $\dot{x}$ being $=1$) $y\dot{y}$, we have C H $=\overline{2ay-y^2}|^{\frac{1}{2}}$ and by making y vaniſh it becomes $=0$, ſo that the Vertices of the evolute and involute Curves coincide.

EXAMPLE IV.

70. *To find the Radius of Curvature at any Point* B *of the Curve* A D, *whoſe Nature is ſuch, that the Triangle* C B T, *made of the Ordinate,*

dinate, Tangent, and Subtangent, is always proportional to the Ordinate C B, *or, whoſe Subtangent* C T *is always the ſame.*



PUT the given Subtangent C T $=a$, G C $=x$, C B $=y$; then by *Art.* 35. $\frac{\dot{x}y}{\dot{y}}=a$, *i. e.* (if $\dot{x}$ be made $=1$,) $\frac{y}{\dot{y}}=a$ ∴ $\dot{y}=\frac{y}{a}$, therefore $\dot{y}^2=\frac{y^2}{a^2}$ and, (becauſe here y flows with an accelerated Motion, or, its ſecond Fluxion is Affirmative,) $\ddot{y}=\frac{\dot{y}}{a}$, *i. e.* (by ſubſtituting $\frac{y}{a}$ for $\dot{y}$ its equal,) $\ddot{y}=\frac{y}{a^2}$: Now by ſubſtituting for $\dot{y}^2$ and $\ddot{y}$ theſe their Values in $\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{-\ddot{y}}$ (*Art.* 64.)

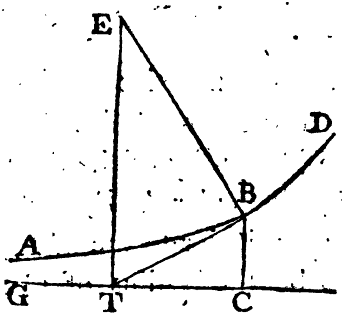
64.) we ſhall have $\frac{\overline{1+\frac{y^2}{a^2}}|^{\frac{3}{2}}\times a^2}{-y}=\frac{\overline{a^2+y^2}|^{\frac{3}{2}}}{-ay}=$ the Radius of Curvature ſought: where the negative Sine ſhews only that the Curve is Convex towards the Axis G E, and that therefore the Radius of Evolution muſt lie on the other Side of the Curve.

Note, The above Curve is the logarithmic Curve, whoſe Aſſymptote is E G; ſo called, becauſe when the ſaid Aſſymptote is divided into any Number of equal Parts, as in the Points G, T, C, E, the Ordinates to theſe Points will be in geometrical Progreſſion, *i. e.* G T, G C, *&c.* will be the Logarithms of the Ordinates T F, C B, *&c.*

EXAMPLE V.

71. *To find the Radius of Curvature for any Point* B *of the Curve* A D, *whoſe Nature is ſuch, that the Tangent* B T *is every where æqual to the ſame given Line* $=a$.

PUT G C $=x$, C B $=$, then, by 47 *E.* 1, $\overline{\overline{TB}|^2-\overline{BC}|^2}|^{\frac{1}{2}}=$ C T, *i. e.* $\overline{a^2-y^2}|^{\frac{1}{2}}=$ C T $=$ (by

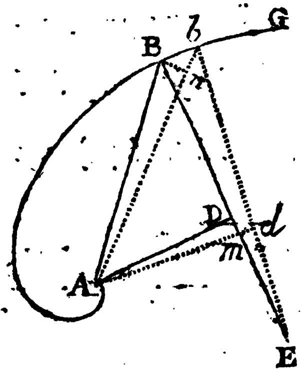


(by *Art.* 35.) $\frac{\dot{x}y}{\dot{y}}$ or, making $\dot{x}=1$, $\overline{a^2-y^2}|^{\frac{1}{2}}=\frac{y}{\dot{y}}$ $\therefore \dot{y}=\frac{y}{\overline{a^2-y^2}|^{\frac{1}{2}}}$, and $\dot{y}^2=\frac{y^2}{a^2-y^2}$, and ſuppoſing the Fluxion of $\dot{y}$ affirmative, $\ddot{y}=\frac{\dot{y}\times\overline{a^2-y^2}|^{\frac{1}{2}}+\frac{y^2\dot{y}}{\overline{a^2-y^2}|^{\frac{1}{2}}}}{a^2-y^2}$, *i. e.* by ſubſtituting for $\dot{y}$ its equal, $\ddot{y}=\frac{y+\frac{y^3}{a^2-y^2}}{a^2-y^2}=\frac{a^2y}{\overline{a^2-y^2}|^2}$. Now by writing for $\dot{y}^2$ and $\ddot{y}$ theſe their Values in $\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{-\ddot{y}}$ (*ſee Art.* 64.) we ſhall have $\frac{\overline{1+\frac{y^2}{a^2-y^2}}|^{\frac{3}{2}}\times\overline{a^2-y^2}|^2}{-a^2y}$, *i. e.* $\frac{\overline{a^2}|^{\frac{3}{2}}\times\overline{a^2-y^2}|^{\frac{1}{2}}}{-a^2y}$ or

—a

$-\frac{a}{y}\times\overline{a^2-y^2}|^{\frac{1}{2}}=$ the Radius Curvature required: Where the negative Sign only ſhews its Poſition. Hence the following Conſtruction: On the Extremity of the Subtangent T, erect the Perpendicular T E, and draw the Line B E perpendicular to the Tangent T B; then will B E be the Radius of Curvature at the Point B, or, the Point E will be in the evolute Curve, for the Triangles C B T and B T E will be Similar, and conſequently B C : C T : : T B : B E, *i. e.* $y:\overline{a^2-y^2}|^{\frac{1}{2}}::a:\frac{a}{y}\times\overline{a^2-y^2}|^{\frac{1}{2}}=$ B E.

The general Expreſſion for the Radius of Evolution or Curvature, found *Art.* 62. being only for Curves referred to an Axis; we ſhall now deduce one for Spirals, or thoſe referred to a fixt or central Point.



72. LET ABG be the Curve; A the central Point, or that from which all the Ordinates iſſue; and BE, the Radius of Curvature at the Point B, *i. e.* let E be ſuppoſed in the evolute Curve: Conceive A*b* and E*b* indefinitely near to AB and EB; and AD, A*d*, perpendicular to EB, E*b*; then will the Points D and *m* be indefinitely near to a Coincidence, and B*m* and A*m* may be taken as equal to BD and AD, the Difference being indefinitely ſmall. Now if with the Radius AB, the little circular Arch B*n* be deſcribed, (which may be conſidered as a little right Line, as may alſo the Curve B*b*,) the little right-angled Triangle *bn*B will be ſimilar to the right angled Triangle BDA: (For $\angle ABn = \angle EBb$ therefore, $\angle EBn$ being common, $\angle ABD = \angle nBb$, and therefore, the Angles at D and *n* being right, $\angle BAD = \angle Bbn$:) therefore $bB : Bn :: AB : BD$, that is, (if we put $AB = y$, $Bn = x'$, $nb = y'$) becauſe by 47 *E.* 1.

$$\overline{x'^2+y'^2}\big|^{\frac{1}{2}} = Bb,\ \overline{x'^2+y'^2}\big|^{\frac{1}{2}} : x' :: y : \frac{x'y}{\overline{x'^2+y'^2}\big|^{\frac{1}{2}}}$$

$=$ BD, or B*m*; or, ſubſtituting $\dot{x}$ for x' and $\dot{y}$ for y', $\frac{\dot{x}y}{\overline{\dot{x}^2+\dot{y}^2}\big|^{\frac{1}{2}}} =$ BD or B*m*. Again B*b* : *bn* :: BA : AD, that is, $\overline{x'^2+y'^2}\big|^{\frac{1}{2}} : y' :: y :$

$\frac{yy'}{\overline{x'^2+y'^2}|^{\frac{1}{2}}}=$ A D or A *m*, or $\frac{y\dot{y}}{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}}=$ A D or A*m*, whoſe Fluxion, (ſuppoſing $\dot{x}$ invariable,) is

$$\frac{\overline{\dot{y}^2+y\ddot{y}}\times\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}-\frac{\dot{y}\ddot{y}\times y\dot{y}}{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}}}{\dot{x}^2+\dot{y}^2}=\frac{\dot{x}^2\dot{y}^2+\dot{y}^4+y\dot{x}^2\ddot{y}}{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}}$$

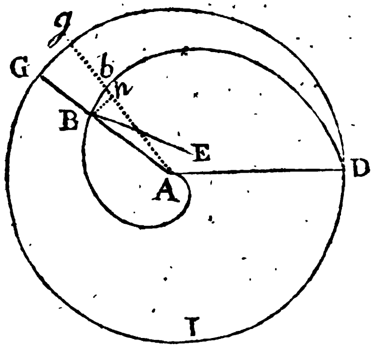
$=md.$ But by the ſimilar Triangles E B *b* and E *m d*, we have B *b*—*m d* : B *b* : : B E— — E *m*, or, *m* B : B E ; or, $\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}-\frac{\dot{x}^2\dot{y}^2-\dot{y}^4-y\dot{x}^2\ddot{y}}{\overline{\dot{x}^4+\dot{y}^2}|^{\frac{3}{2}}}=\frac{\dot{x}^4+\dot{x}^2\dot{y}^2-y\dot{x}^2\ddot{y}}{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}}:\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}::\frac{\dot{x}y}{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}}:\frac{y\times\overline{\dot{x}^2\times\dot{y}^2}|^{\frac{1}{2}}}{\dot{x}^3+\dot{x}\dot{y}^2-y\dot{x}\ddot{y}}=$ B E, which is a general Expreſſion for the Radius of Evolution, or Curvature, of all Curves referred to a fixt or central Point, when $\dot{x}$ is invariable. Wherefore,

73. If $\dot{x}$ be made $=1$, the general Expreſſion for B E, the Radius of Curvature, will be $=\frac{y\times\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{1+\dot{y}^2-y\ddot{y}}$, and if we put the Equation of the given Spiral into Fluxions, making $\dot{x}=1$, and put this reſulting Equation into Fluxions again, the Values of the ſecond and Square of the firſt Fluxions of y being determined, and ſubſtituted for them in this general Expreſſion, will

give the Radius of Evolution or Curvature required, as in the following Examples.

EXAMPLE I.

74. *To find the Radius of Curvature at any Point* B *of the Spiral of* Archimedes ABD. *(See Art.* 50.*)*



Put the Circumference of the generating Circle DIGD $= a$, Radius AD $= b$, Ordinate AB $= y$, Arch DIG $= z$, B$n = x'$, $nb = y'$, G$g = z'$, the Point g being ſuppoſed indefinitely near to G. Now by the Nature of the Curve $a : b :: z : y$ or $z = \frac{ay}{b}$ in Fluxions $\dot{z} = \frac{a\dot{y}}{b}$, but $y : x' :: b : z' = \frac{bx'}{y}$, or, $\dot{z} = \frac{b\dot{x}}{y}$

$\therefore \frac{a\dot{y}}{b} = \frac{b\dot{x}}{y}$ or, making $\dot{x} = 1$, $\frac{a\dot{y}}{b} = \frac{b}{y}$ which Equation reduced gives $\dot{y} = \frac{b^2}{ay}$: Now the Fluxion of this again is $\ddot{y} = \frac{-ab^2\dot{y}}{a^2y^2} =$ (by writing $\frac{b^2}{ay}$ for $\dot{y}$) $\frac{-b^4}{a^2y^3}$, and $\dot{y}^2 = \frac{b^4}{a^2y^2}$. And if we ſubſtitute for $\dot{y}^2$ and $\ddot{y}$ theſe their Values we ſhall have $\frac{y \times \overline{1+\dot{y}^2}|^{\frac{3}{2}}}{1+\dot{y}^2-y\ddot{y}}$ (*Art.* 73.) $=$

$$\frac{y \times \overline{1+\frac{b^4}{a^2y^2}}|^{\frac{3}{2}}}{1+\frac{b^4}{a^2y^2}+\frac{b^4}{a^2y^2}} = \frac{\overline{a^2y^2+b^4}|^{\frac{3}{2}}}{a^3y^2+2ab^4} = BE$$

the Radius of Curvature ſought.

EXAMPLE II.

75. *Let the Nature of any Spiral be expreſſed by this Equation* $ay^m = b^mz$, *where m ſtands for any whole Number or Fraction* ad libitum: *To find the Radius of Curvature for any Point.*

THIS Equation in Fluxions is $amy^{m-1}\dot{y} = bm\dot{z}$, *i. e.* $\frac{amy^{m-1}}{b^m}\dot{y} = \dot{z} = \frac{b\dot{x}}{y}$ by the laſt Example, *i. e.* (by making $\dot{x}=1$) $\frac{amy^{m-1}}{b^m}\dot{y}=\frac{b}{y}$ or $\dot{y}=\frac{b^{m+1}}{amy^m}$. Now, this put into Fluxions again is, $\ddot{y}=\frac{-am^2y^{m-1}\times\dot{y}b^{m+1}}{a^2m^2y^{2m}}=\frac{-b^{m+1}\dot{y}}{ay^{m+1}}=$ (by ſubſtituting for $\dot{y}$, its equal) $\frac{-b^{2m+2}}{a^2my^{2m+1}}$; and $\dot{y}^2=\frac{b^{2m+2}}{a^2m^2y^{2m}}$ and if we write for $\dot{y}^2$ and $\ddot{y}$, theſe their Values, in $\frac{y\times\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{1+\dot{y}^2-y\ddot{y}}$ (*Art.* 73.) we ſhall have

$$\frac{y\times\overline{1+\frac{b^{2m+2}}{a^2m^2y^{2m}}}\Big|^{\frac{3}{2}}}{1+\frac{b^{2m+2}}{a^2m^2y^{2m}}+\frac{b^{2m+2}}{a^2my^{2m}}},\ i.\ e.$$

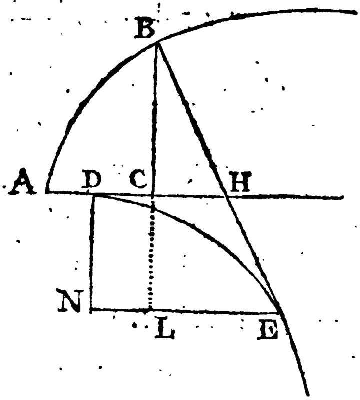
$$\frac{\overline{a^2m^2y^{2m}+b^{2m+2}}|^{\frac{3}{2}}}{a^3m^3y^{3m-1}+\overline{m+1}\,.\,amy^{m-1}b^{2m+2}}=$$ the Radius of Curvature ſought; which, when $m=1$, is $=\frac{a^2y^2+b^4}{a^3y^2+2ab^4}$, as before.

CHAP.

CHAP. VII.

Of finding the Nature of the Evolutes of given Involute CURVES.

AS it is abſolutely neceſſary for the Learner to be well acquainted with the foregoing Chapter, before he enters upon this, we ſhall not here define the Meaning of Evolute and Involute Curves, it being ſufficiently explained in that.



76. LET BE, be the Radius of Evolution at any Point B of the involute Curve AB, whoſe Abſciſs is AC $= x$, and Ordinate CB $= y$, parallel to HA draw EN; produce BC to L, and equal and parallel to CL draw DN;

then will the Triangles B H C and B E L be ſimilar; and therefore B H : H C :: B E : E L, *i. e.* (*Art.* 62.) $\frac{y}{\dot{x}} \times \overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}} : \frac{y\dot{y}}{\dot{x}} :: \frac{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}}{\ddot{y}} : \frac{\dot{y}\times\overline{\dot{x}^2+\dot{y}^2}}{\ddot{y}} =$ E L, and C H : C B :: E L : L B, *i. e.* $\frac{y\dot{y}}{\dot{x}} : y :: \frac{\dot{y}\times\overline{\dot{x}^2+\dot{y}^2}}{\ddot{y}} : \frac{\dot{x}\times\overline{\dot{x}^2+\dot{y}^2}}{\ddot{y}} =$ L B, and theſe are generally Expreſſions for E L, and L B, when $\dot{x}$ is conſidered as invariable, and the Fluxion of $\dot{y}$ as Negative. Hence

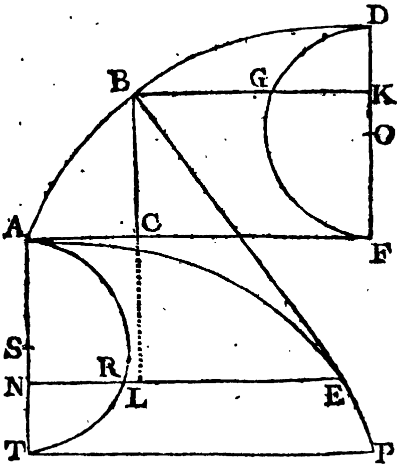
77. If $\dot{x}=1$, and the Fluxion of $\dot{y}$ be negative, the general Expreſſion for E L will be $= \frac{\dot{y}\times\overline{1+\dot{y}^2}}{\ddot{y}}$, and that for L B $= \frac{1+\dot{y}^2}{\ddot{y}}$. Now by help of the Equation of the Curve, exterminate $\dot{y}$, $\dot{y}^2$, $\ddot{y}$, out of theſe Expreſſions, as in the preceding Chapter; and by *Art.* 65. find the vertical Diſtance A D; then, if we put the Abſciſs of the Evolute D N $=u$ and Ordinate N E $=v$; by help of theſe two Equations $u=$ B L—B C, and $v=$ A C—A D+L E, we may get the Nature of the evolute Curve D E required.

78. *Note*, If the given Involute be Convex towards its Axis and the Fluxions of *x* and *y* increaſe together, the 2d Fluxion of *y* will be affirmative, and the general Expreſſions for E L and L B

will be $\frac{\dot{y}\times\overline{1+\dot{y}^2}}{-\ddot{y}}$ and $\frac{1+\dot{y}^2}{-\ddot{y}}$; where the Negative Sign only ſhews that the Points E and L muſt be taken on the concave ſide of the Curve, that is, on the other ſide of the Curve with regard to x and y.

EXAMPLE I.

79. *To find the Nature of that Curve by whoſe Evolution the Cycloid* A B D *is deſcribed.*



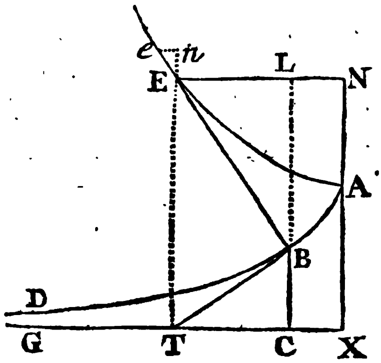
PUT A C$=x$, CB$=y$, Arch F G$=z$, O D or OF$=a$; then (*Art.* 69.) $\dot{y}=\frac{\overline{2ay-y^2}|^{\frac{1}{2}}}{y}$, $\dot{y}^2=$

$2ay-y^2$

$\frac{2ay-y^2}{y^2}$, and $\ddot{y}=\frac{a}{y^2}$ wherefore (*ſee Art.* 77.) $BL = \frac{1+\dot{y}^2}{\ddot{y}} = \frac{\overline{1+\frac{2ay-y^2}{y^2}} \times y^2}{a} = 2y$, and $LE = \frac{\dot{y}\times\overline{1+\dot{y}^2}}{\ddot{y}} = \dot{y}\times 2y = 2\times\overline{2ay-y^2}|^{\frac{1}{2}}$ Hence, if we put the Abſciſs $AN = u$, and Ordinate $NB = v$, we have $u = 2y - y = y$, and $v = x + 2\times\overline{2ay-y^2}|^{\frac{1}{2}}$, *i. e.* (becauſe $x = z - \overline{2ay-y^2}|^{\frac{1}{2}}$, *Art.* 69.) $v = z + \overline{2ay-y^2}|^{\frac{1}{2}}$, or, writing u for y its equal, $v = z + \overline{2au-u^2}|^{\frac{1}{2}}$, Wherefore the evolute Curve A E Q is a Cycloid, and equal to the given Cycloid A B D: For let $AS = ST = a$, then (A N being = F K,) $AR = FG = z$, and $NR = \overline{2au-u^2}|^{\frac{1}{2}} = KG$, and therefore $AR + RN = z + \overline{2au-u^2}|^{\frac{1}{2}}$; *i. e.* $AR + RN = NE$, which is the Property of the Cycloid, therefore A E P is a Cycloid; and becauſe $AT = FD$, therefore the Cycloids A E P and A B D are equal.

EXAMPLE II.

80. *To find the Nature of the Evolute of the Curve* A D, *whoſe Tangent* B T, *is every where equal to the ſame given Line* $= a$.

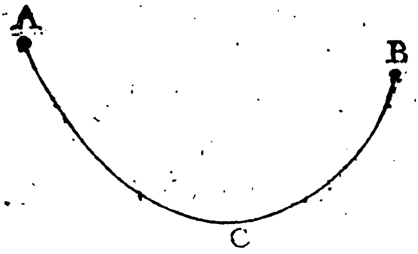


LET BE be the Radius of Evolution at the Point B; then by *Art.* 71. if a Perpendicular be erected on the Point T, it will paſs thro' the Point E: wherefore when the Point T coincides with X, that is, when the Tangent and Ordinate become equal, or the Points B and A coincide; the Point E will likewiſe coincide with A, conſequently the Vertex of the Evolute coincides with that of the Involute. Put $XC = x$, $CB = y$, $AN = u$, $NE = v$, then (*Art.* 71.) $\dot{y} = \frac{y}{\overline{a^2 - y^2}|^{\frac{1}{2}}}$, $\dot{y}^2 = \frac{y^2}{a^2 - y^2}$, and the Fluxion of $\dot{y}$ being Negative, $-\ddot{y} = \frac{a^2 y}{\overline{a^2 - y^2}|^2}$. Wherefore BL (*Art.* 77.) =

$1 + \dot{y}$

$\frac{1+\dot{y}^2}{\ddot{y}} = \frac{1+\frac{y^2}{a^2-y^2}}{-\frac{a^2y}{\overline{a^2-y^2}\rvert^2}} = - : \frac{a^2-y^2}{y}$, and LE $=$ $\dot{y} \times \frac{1+\dot{y}^2}{\ddot{y}} = \dot{y} \times \mathrm{BL} = \frac{y}{\overline{a^2-y^2}\rvert^{\frac{1}{2}}} \times - : \frac{a^3-y^2}{y} = -\overline{a^2-y^2}\rvert^{\frac{1}{2}}$, where the negative Sign ſhews that the Points L and E muſt be taken on the concave Side of the Curve. Hence we have $u = \frac{a^2-y^2}{y} + y - a$ which gives $y = \frac{a^2}{u+a}$ and $\dot{y} = -\frac{a^2\dot{u}}{\overline{u+a}\rvert^2}$: alſo $v = x + \overline{a^2-y^2}\rvert^{\frac{1}{2}}$, and $\dot{v} = \dot{x} - \frac{y\dot{y}}{\overline{a^2-y^2}\rvert^{\frac{1}{2}}}$, *i. e.* (becauſe (*Art.* 36.) $-\frac{\dot{x}y}{\dot{y}} = \overline{a^2-y^2}\rvert^{\frac{1}{2}}$ or $\dot{x} = -\frac{\dot{y}}{y} \times \overline{a^2-y^2}\rvert^{\frac{1}{2}}$) $\dot{v} = -\frac{\dot{y}}{y} \times \overline{a^2-y^2}\rvert^{\frac{1}{2}} - \frac{y\dot{y}}{\overline{a^2-y^2}\rvert^{\frac{1}{2}}} = \frac{-a^2\dot{y}}{y \times \overline{a^2-y^2}\rvert^{\frac{1}{2}}}$; or, ſubſtituting for y and $\dot{y}$, their above Values, $\dot{v} = \frac{a\dot{u}}{\overline{u^2+2au}\rvert^{\frac{1}{2}}}$, which is an Equation for the evolute Curve AD, and is an Equation for the catenary Curve; therefore the Evolute AD is the Catenary *.

Note, The above Equation of the Evolute may be found otherwiſe, thus; Let E$n=u'$, $ne=v'$; then, the Triangles enE and EBT being Similar, as $en : nE :: TB : BE$, *i. e.* $v' : u' :: a : \frac{au'}{v'} = BE$ or $\frac{a\dot{u}}{\dot{v}} = BE$: But BE $= \overline{\overline{ET}|^2 - \overline{TB}|^2}|^{\frac{1}{2}}$, *i. e.* (becauſe $ET = XN = u + a$) $BE = \overline{\overline{u+a}|^2 - a^2}|^{\frac{1}{2}} = \overline{u^2 + 2au}|^{\frac{1}{2}}$ therefore $\frac{a\dot{u}}{\dot{v}} = \overline{u^2 + 2au}|^{\frac{1}{2}}$ and $\dot{v} = \frac{a\dot{u}}{\overline{u^2 + 2au}|^{\frac{1}{2}}}$.



[*] The Catenary is a Curve as A C B formed by a flexible Line hanging freely from two Points of Suſpenſion A, B, whether theſe Points be horizontal or not.

81. To find the Evolute of a Spiral, you muſt find the Radius of Curvature at ſeveral Points, which will give as many Points in the evolute Curve; then a curve Line drawn thro' thoſe Points ſo found, will be the Evolute ſought.

CHAP. VIII.

Of infinite SERIES.

IT being abſolutely neceſſary for the Learner to underſtand ſomething of infinite Series, before he can find the Fluents, or flowing Quantities, of Fluxions expreſs'd after a fractional Manner, or wherein there are Surds or radical Quantities; and becauſe the next Chapter treats of the finding the flowing Quantities or Fluents of fluxional Expreſſions; we thought it not improper to add this Chapter, tho' it is, in ſome Meaſure, foreign to our Buſineſs.

PROB. I.

To reduce fractional Quantities into infinite Series.

EXAMPLE I.

82. *Let it be required to reduce* $\frac{b}{a+x}$ *into an infinite Series.*

Place $a+x$ as a Diviſor and b as a Dividend and divide as in common Diviſion, till you have 4, 5, 6, or more Terms in the Quotient; after which (in moſt Caſes) you may find as many Terms as you pleaſe, by conſidering the Law of the Progreſſion of the Terms already found. Thus the four firſt Terms being $\frac{b}{a} - \frac{bx}{a^2} + \frac{bx^2}{a^3} - \frac{bx^3}{a^4}$, the Law of the Continuation of the Diviſion, or Series, is plain; for the Quotient conſiſts of an infinite Series of Terms whoſe Numerators are the Powers of x leſs by 1 than the Number of the Order, multiplied by b; and Denominators the Power of a, whoſe Indices are the Number of the Order of the Terms: And having its Signs changed alternately, thus the fifth Term will be $+\frac{bx^4}{a^5}$ and the ſixth Term $-\frac{bx^5}{a^6}$; and if an infinite Number or Series of Terms be ſo taken, it will be the exact Quotient of the Diviſion, and conſequently exactly equal to the given fractional Expreſſion; but (generally) a few of the firſt Terms of the Series are near enough the Truth for any Purpoſe.

Or, if we put x before a in the Denominator of the above given fractional Expreſſion, *i. e.*

i. e. if the Diviſor be expreſs'd thus $x+a$ inſtead of $a+x$; the Quotient or Series will be $\frac{b}{x}-\frac{ba}{x^2}+\frac{ba^2}{x^3}-\frac{ba^3}{x^4}$, &c. whence the Law of the Continuation of the Series may be obſerved as before.

83. *Note.* There will always be as many Quotients or infinite Series, as there are Terms in the Denominator or Diviſor, though only one true; and to find this, you muſt always place the greateſt Terms in the Diviſor and Dividend firſt, *i. e.* if, in the laſt Example, for Inſtance, a be greater than x, then a muſt be the firſt Term in the Diviſor, and $\frac{b}{a}-\frac{bx}{a^2}+\frac{bx^2}{a^3}-\frac{bx^3}{a^4}$, &c. will be the true Series; but if x be greater than a, then x muſt be the firſt Term in the Diviſor, and $\frac{b}{x}-\frac{ba}{x^2}+\frac{ba^2}{x^3}-\frac{ba^3}{x^4}$, &c. will be the true Series; the other, then being a diverging one, and conſequently the further you go in the Series, the farther it will be from the Truth.

EXAMPLE II.

84. *Let it be required to throw* $\frac{a^2}{a^2+2ax+x^2}$ *into an infinite Series, ſuppoſing* a *to be greater than* x.

OPERATION.

$$a^2+2ax+x^2)\ a^2 \ldots\ldots \left(1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}\right.$$

$$\underline{a^2+2ax+x^2}$$

$$0-2ax-x^2$$

$$\underline{-2ax-4x^2-\frac{2x^3}{a}}$$

$$0+3x^2+\frac{2x^3}{a}$$

$$\underline{3x^2+\frac{6x^3}{a}+\frac{3x^4}{a^2}}$$

$$0-\frac{4x^3}{a}-\frac{3x^4}{a^2}$$

$$\underline{-\frac{4x^3}{a}-\frac{8x^4}{a^2}-\frac{4x^5}{a^3}}$$

$$0+\frac{5x^4}{a^2}+\frac{4x^5}{a^3}$$

Now from theſe four Terms, it is eaſy to ſee that the Law of Continuation is ſuch, that the Numerators are the Powers of x, whoſe Exponents are 1 leſs than the Number of the Order, multiplied into the ſaid Number; and the Denominators, the Powers of a, whoſe Exponents are the ſame with thoſe of the correſpondent Numerators; the Signs being changed alternately: So that the fifth Term is $+\frac{5x^4}{a^4}$, the ſixth Term $-\frac{6x^5}{a^5}$; and ſo on.

PROB. II.

To reduce a compound ſurd Quantity into an infinite Series, *i. e.* to free a compound Expreſſion from Surds, by throwing it into an infinite Series.

EXAMPLE I.

85. *Let it be required to throw* $\sqrt{a^2+x^2}$ *or* $\overline{a^2+x^2}|^{\frac{1}{2}}$ *into an infinite Series.*

Note. In Order to have a true Series, the greateſt Term muſt be always plac'd firſt, as in *Prob.* 1. Therefore, ſuppoſing a^2 to be greater than x^2,

Take the Square Root of a^2, which is a, for the firſt Term in the Root (ſee the Operation below,) then this ſquar'd and ſubtracted from a^2+x^2 leaves $+x^2$. And this Remainder divided (as in the common Extraction of the Square Root) by $2a$, *viz.* the double of the firſt Term, gives $\frac{x^2}{2a}$ for the ſecond Term in the Root, which, together with the double of the firſt Term, being multiplied by $\frac{x^2}{2a}$ the ſaid ſecond Term, gives $x^2+\frac{x^4}{4a^2}$ and this ſubtracted from x^2 leaves $-\frac{x^4}{4a^2}$ which divided by the double of the two 1ſt Terms in the Root, *viz.* by $2a+\frac{x^2}{a}$ gives $-\frac{x^4}{8a^3}$ for the 3d Term in the Root which, together with the double of the two firſt Terms, *viz.* $2a+\frac{x^2}{a}$. multiplied by $-\frac{x^4}{8a^3}$ the ſaid third Term, gives $-\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6}$ and this ſubtracted from $-\frac{x^4}{4a^2}$, leaves $\frac{x^6}{8a^4}-\frac{x^8}{64a^6}$, which divided by the double of the three Terms of the Root already found, *viz.* by $2a+\frac{x^2}{a}-\frac{x^4}{4a^3}$ gives $\frac{x^6}{16a^5}$ for the fourth

Term in the Root, and after the ſame Manner may be found any Number of Terms in the Root; and after the Law of the Progreſſion or Continuation is diſcovered, the Series may be continued on at Pleaſure.

OPERATION.

$$a^2+x^2\left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}\right.$$

$$\begin{array}{rl}
 & a^2 \\
2a)\ldots & \overline{0+x^2} \\
 & +x^2+\frac{x^4}{4a^2} \\
2a+\frac{x^2}{a})\ldots\ldots & \overline{0-\frac{x^4}{4a^2}} \\
 & -\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x_8}{64a^6} \\
2a+\frac{x^2}{a}-\frac{x^4}{4a^3})\ldots\ldots & \overline{0+\frac{x^6}{8a^4}-\frac{x^8}{64a^6}}
\end{array}$$

EXAMPLE II.

86. *Let it be required to throw* $\overline{x-x^2}|^{\frac{1}{2}}$ *into an infinite Series; where x is ſuppoſed to be be leſs than Unity, or x greater than* x^2.

OPERATION.

$$
\begin{array}{rl}
 & x-x^2\ \left(x^{\frac{1}{2}}-\frac{1}{2}x^{\frac{3}{2}}-\frac{1}{8}x^{\frac{5}{2}}-\frac{1}{16}x^{\frac{7}{2}}-\frac{5}{128}x^{\frac{9}{2}}\right. \\
 & \underline{x\qquad\quad} \\
2x^{\frac{1}{2}}) & 0-x^2 \\
 & \underline{-x^2+\frac{1}{4}x^3} \\
2x^{\frac{1}{2}}-x^{\frac{3}{2}}) & \ldots 0-\frac{1}{4}x^3 \\
 & \underline{\qquad -\frac{1}{4}x^3+\frac{1}{8}x^4+\frac{1}{64}x^5} \\
2x^{\frac{1}{2}}-x^{\frac{3}{2}}-\frac{1}{4}x^{\frac{5}{2}}) & \ldots\ldots 0\quad -\frac{1}{8}x^4-\frac{1}{64}x^5 \\
 & \underline{\qquad\qquad -\frac{1}{8}x^4+\frac{1}{16}x^5+\frac{1}{64}x^6+\frac{1}{256}x^7} \\
2x^{\frac{1}{2}}-x^{\frac{3}{2}}-\frac{1}{4}x^{\frac{5}{2}}-\frac{1}{8}x^{\frac{7}{2}}) & \ldots\ldots\ldots 0\quad -\frac{5}{64}x^5-\frac{1}{64}x^6-\frac{1}{256}x^7 \\
 & \qquad\qquad\qquad\&c.
\end{array}
$$

So that $\overline{x-x^2}|^{\frac{1}{2}}$ is $=x^{\frac{1}{2}}-\frac{1}{2}x^{\frac{3}{2}}-\frac{1}{8}x^{\frac{5}{2}}-\frac{1}{16}x^{\frac{7}{2}}-\frac{5}{128}x^{\frac{9}{2}}$, &c.

And after the ſame Manner may any common ſurd Quantity be reduced into an infinite Series. But with much greater Eaſe and Expedition,

87. All manner of fractional and ſurd Quantities, may be reduced into infinite Series, by a moſt curious and excellent Theorem invented for that Purpoſe, by the Great and Illuſtrious Sir *Iſaac Newton*, called his

UNIVERSAL THEOREM,

which is this, $\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \frac{m-4n}{5n}EQ +$ &c. wherein it muſt be obſerved that, $P+PQ$ repreſents the Quantity, whoſe Root, Dimenſion, or Root of the Dimenſion, is required to be thrown into an infinite Series; P the firſt Term of that Quantity, which muſt be the greateſt; Q the reſt of the Terms divided by the firſt; $\frac{m}{n}$ the numerical Index of the Dimenſion or Power of $P+PQ$, whether that Power be Affirmative or Negative, Integral or Fractional; and A, B, C, D, E, &c. the Terms in the Series or Quotient already found, *i. e.* $A=P^{\frac{m}{n}}$, $B=\frac{m}{n}AQ$, $C=\frac{m-n}{2n}BQ$

$D=$

$D=\frac{m-2n}{3n}CQ$, $E=\frac{m-3n}{4n}DQ$, or A = the firſt Term, B = the ſecond Term, C = the third Term, D = the fourth Term, E = the fifth Term, &c. —— The following Examples will explain, and ſhew the great Uſe of this curious Theorem.

EXAMPLE I.

88. *Let it be required to throw* $\overline{a^2+x^2}|^{\frac{1}{2}}$ *into an infinite Series,* a^2 *being greater than* x^2.

HERE $P=a^2$, $Q=\frac{x^2}{a^2}$, $m=1$, $n=2$, $A=P^{\frac{m}{n}}=a$, $B=\frac{m}{n}AQ=\frac{x^2}{2a}$, $C=\frac{m-n}{2n}BQ=-\frac{x^4}{8a^3}$, $D=\frac{m-2n}{3n}CQ=\frac{x^6}{16a^5}$, $E=\frac{m-3n}{4n}DQ=\frac{5x^8}{128a^7}$, &c. therefore $\overline{a^2+x^2}|^{\frac{1}{2}}=a+\frac{x^2}{2a}$ $\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}$, &c.

Ex-

EXAMPLE II.

89. *Required to put* $\overline{x-x^2}|^{\frac{1}{2}}$ *into an infinite Series; x being greater than* x^2.

Here $P=x$, $Q=-\frac{x^2}{x}=-x$, $m=1$, $n=2$, $A=x^{\frac{1}{2}}$, $B=-\frac{1}{2}x^{\frac{3}{2}}$, $C=-\frac{1}{8}x^{\frac{5}{2}}$, $D=-\frac{1}{16}x^{\frac{7}{2}}$, &c. therefore $\overline{x-x}|^{\frac{1}{2}}=x^{\frac{1}{2}}-\frac{1}{2}x^{\frac{3}{2}}-\frac{1}{8}x^{\frac{5}{2}}-\frac{1}{16}x^{\frac{7}{2}}$, &c.

EXAMPLE III.

90. *Required to put* $\frac{b}{a+x}$ *into an infinite Series; x being less than a.*

$\frac{b}{a+b}$ is $=b\times\frac{1}{a+x}=b\times\overline{a+x}|^{-1}$, $P=a$, $Q=\frac{x}{a}$, $m=-1$, $n=1$, $A=a^{-1}=\frac{1}{a}$, $B=-\frac{x}{a^2}$, $C=\frac{x^2}{a^3}$, $D=-\frac{x^3}{a^4}$, $E=\frac{x^4}{a^5}$, &c. therefore $b\times\overline{a+x}|^{-1}=b\times:\frac{1}{a}-\frac{x}{a^2}+\frac{x^2}{a^3}-\frac{x^3}{a^4}+\frac{x^4}{a^5}$, &c. that is, $\frac{b}{a+x}=\frac{b}{a}-\frac{bx}{a^2}+\frac{bx^2}{a^3}-\frac{bx^3}{a^4}+\frac{bx^4}{a^5}$, &c.

EXAMPLE IV.

91. *Required to put* $\frac{a^2}{\overline{a+x}|^2}$ *into an infinite Series; where a is greater than x.*

$\frac{a^2}{\overline{a+x}|^2}$ is $= a^2 \times \overline{a+x}|^{-2}$, $P=a$, $Q=\frac{x}{a}$, $m=-2$, $n=1$, $A=a^{-2}=\frac{1}{a^2}$, $B=-\frac{2x}{a^3}$, $C=\frac{3x^2}{a^4}$, $D=-\frac{4x^3}{a^5}$, $E=\frac{5x^4}{a^6}$, &c. therefore $a^2 \times \overline{a+x}|^{-2} = a^2 \times : \frac{1}{a^2} - \frac{2x}{a^3} + \frac{3x^2}{a^4} - \frac{4x^3}{a^5} + \frac{5x^4}{a^6}$ &c. that is, $\frac{a^2}{\overline{a+x}|^2} = 1 - \frac{2x}{a} + \frac{3x^2}{a^2} - \frac{4x^3}{a^3} + \frac{5x^4}{a^4}$, &c.

29. But we may find the Series anſwering to any propoſed Quantity, by the following *univerſal Theorem*, viz. $\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} \times : 1 + \frac{m}{n}Q + \frac{m}{n} \times \frac{m-1}{2n} Q^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} Q^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} Q^4 +$ &c. (which indeed is the ſame as the other, tho' differently expreſſed) with ſtill more Eaſe and

Expedition; and that without any previous Deduction; if we conſider that both the Numerators and Denominators of the Fractions $\frac{m}{n}$, $\frac{m-n}{2n}$, $\frac{m-2n}{3n}$, $\frac{m-3n}{4n}$, &c. are Series of Numbers in arithmetical Progreſſion, which have the ſame common Difference n. This will appear by the following Examples.

EXAMPLE I.

93. *Let* $\overline{x-x^2}|^{\frac{1}{2}}$ *be required to be put into an infinite Series.*

Here $P=x$, $Q=-\frac{x^2}{x}=-x$, $m=1$, $n=2$, therefore $\overline{x-x^2}|^{\frac{1}{2}}=x^{\frac{1}{2}}\times : 1-\frac{1}{2}x+\frac{1\times-1}{2\times4}x^2-\frac{1\times-1\times-3}{2\times4\times6}x^3+\frac{1\times-1\times-3\times-5}{2\times4\times6\times8}x^4-$ &c. *i. e.* $=x^{\frac{1}{2}}\times$; $1-\frac{1}{2}x-\frac{1}{2.4}x^2-\frac{3}{2.4.6}x^3-\frac{3.5}{2.4.6.8}x^4-$ &c. or $x^{\frac{1}{2}}-\frac{x^{\frac{3}{2}}}{2}-\frac{x^{\frac{5}{2}}}{2.4}-\frac{3.x^{\frac{7}{2}}}{2.4.6}-\frac{3.5.x^{\frac{9}{2}}}{2.4.6.8}-$ &c.

EXAMPLE II.

94. *Let* $\overline{a+x}|^{\frac{3}{5}}$ *be put into an infinite Series.*

Here $P = a$, $Q = \frac{x}{a}$, $m = 3$, $n = 5$, therefore $\overline{a+x}|^{\frac{3}{5}} = a^{\frac{3}{5}} \times : 1 + \frac{3.x}{5.a} + \frac{3.-2.x^2}{5.10.a^2} + \frac{3.-2.-7.x^3}{5.10.15.a^3} + \frac{3.-2.-7.-12.x^4}{5.10.15.20.a^4} +$ &c, *i. e.* $= a^{\frac{3}{5}} + \frac{3.x}{5.a^{\frac{2}{5}}} - \frac{6.x^2}{5.10.a^{\frac{7}{5}}} + \frac{6.7.x^3}{5.10.15.a^{\frac{12}{5}}} - \frac{6.7.12.x^4}{5.10.15.20.a^{\frac{17}{5}}}$, &c.

CHAP. IX.

Of finding the Fluent of a given Fluxion.

95. AS it is the Buſineſs of the direct Method of Fluxions to find the Fluxion of any given variable Quantity or Fluent, or the Velocity with which it flows at any particular Point; So it is the Buſineſs of this Chapter or of the inverſe Method of Fluxions, to find or determine the variable Quantity or Fluent when that Velocity or Fluxion is given alone: and this may be done by the following Rules.

96. RULE 1. To find the Fluent of a ſimple Fluxion: Subſtitute the flowing Term for its Fluxion, and it will give the Fluent requir'd. Thus the Fluent of $a\dot{x}$ is ax.

97. RULE 2. To find the Fluent of a fluxionary Expreſſion which is compounded of different Simple ones connected together with the Signs + and — : Find the Fluent of each ſimple Expreſſion by *Rule* 1. which connect together with the Signs of their reſpective Fluxions, and it will be the Fluent ſought. Thus the Fluent of $\dot{x}+\dot{y}-b\dot{z}$ is $x+y-bz$: And the Fluent of $a^2\dot{x}-b^3\dot{y}+\dot{z}$ is a^2x-b^3y+z.

98. RULE 3. To find the Fluent of a fluxional Expreſſion which conſiſts of the Products of two or more variable Quantities drawn into their Fluxions *i. e.* which conſiſts of the Fluxion of each variable Quantity multiplied into the other or Product of the others, as $\dot{x}y+x\dot{y}$ or $\dot{x}yz+x\dot{y}z+xy\dot{z}$; Multiply the flowing Quantities together, and their Rectangle is the Fluent ſought. Thus the Fluent of $\dot{x}y+x\dot{y}$ is xy, the Fluent of $\dot{x}yz+x\dot{y}z+xy\dot{z}$ is xyz.

99. RULE 4. To find the Fluent of a fluxional Expreſſion which conſiſts of the Fluxion of any variable Quantity drawn into any Power of that Quantity, contain'd any Number of times, as $2x^2\dot{x}$; Strike out the fluxional Letter, increaſe the Index of the Power of the variable Quantity by 1, and divide the Coefficient by the Index thus increas'd; and it

will be the Fluent ſought. Thus the Fluent of $2x^2\dot{x}$ is $\frac{2}{3}x^3$, the Fluent of $-\frac{\dot{x}}{x^2}$, *i. e.* of $-x^{-2}\dot{x}$ is $-x^{-2+1}$ divided by $-2+1$, *i. e.* $\frac{-x^{-1}}{-1}$ or x^{-1} or $\frac{1}{x}$, and univerſally the Fluent of $\frac{m}{n}x^{\frac{m}{n}-1}\dot{x}$ is $x^{\frac{m}{n}}$ or of $ax^{\frac{m}{n}}\dot{x}$ is $\frac{an}{m+n}x^{\frac{m+n}{n}}$.

100. RULE 5. To find the Fluent of a compounded fluxional Expreſſion like $\frac{b}{a+x}\dot{x}$; Throw the Expreſſion into an infinite Series (by *Chap.* 8.) and find the Fluent of the Series by the foregoing Rules, and it will be the Fluent ſought. Thus to find the Fluent of $\frac{b}{a+x}\dot{x}$, I throw it into a Series, which (*Art.* 90.) is $\frac{b}{a}\dot{x}-\frac{bx}{a^2}\dot{x}+\frac{bx^2}{a^3}\dot{x}-\frac{bx^3}{a^4}\dot{x}+\frac{bx^4}{a^5}\dot{x}$ &c. and then find the Fluent of this Series, which (*Art.* 100.) is $\frac{b}{a}x-\frac{bx^2}{2a^2}+\frac{bx^3}{3a^3}-\frac{bx^4}{4a^4}+\frac{bx^5}{5a^5}$, &c. and this is the Fluent of $\frac{b}{a+x}\dot{x}$ required. Again to find the Fluent of $\overline{x-x^2}|^{\frac{1}{2}}\dot{x}$ the Expreſſion muſt be thrown into a Series, which

(*Art.* 89) is $x^{\frac{1}{2}}\dot{x}-\frac{1}{2}x^{\frac{3}{2}}\dot{x}-\frac{1}{8}x^{\frac{5}{2}}\dot{x}-\frac{1}{16}x^{\frac{7}{2}}\dot{x}$, &c. and the Fluent of this Series (*Art.* 99.) is $\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{5}x^{\frac{5}{2}}-\frac{1}{28}x^{\frac{7}{2}}-\frac{1}{72}x^{\frac{9}{2}}$, &c. which is the Fluent of $\overline{x-x^2}|^{\frac{1}{2}}\dot{x}$ required.

SCHOLIUM.

101. This Chapter being only the Reverſe of the Second, we thought it needleſs to Demonſtrate the Rules here delivered. But tho' we can find the Fluxion of any Fluent, be the variable Quantity ever ſo much compounded with invariable ones; yet the Fluent of ſuch compounded fluxional Expreſſion, cannot always be had in finite Terms. And tho' no Fluent can have more than one Fluxion, yet a Fluxion may have an infinite Number of Fluents; thus, for Example, the Fluent of $\dot{x}$ may be x or $x\pm a$, where a repreſents any invariable Quantity whatſoever: and to find a, when it muſt be added to, or taken from, the Fluent x, is call'd correcting the Fluent; but this, in general being very difficult to determine, we ſhall carefully avoid, in the following Chapters the giving of Examples in which the Fluents of the Fluxions need ſuch Correction.

CHAP. X.

Of finding the Length of a curve Line.

102. IN Curves referr'd to an Axis (*Fig.* 1.) let B C be perpendicular to C A, *b c* indefinitely near parallel to B C, and *B n*

Fig. 1.

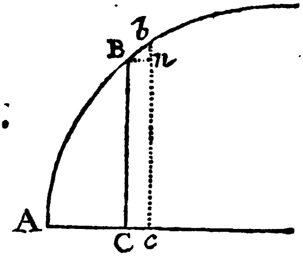
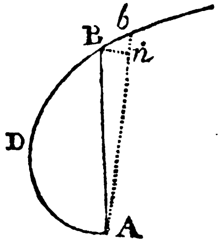


Fig. 2.



equal and parellel to *C c*. And in Curves referr'd to a fixt or central Point (*Fig.* 2.) let *b*A be ſuppos'd indefinitely near to B A, and the little circular Arch B *n* be deſcrib'd with the Radius AB. Put A C. (*Fig.* 1.) $=x$, C B$=y$, Curve AB$=z$; B$n=x'$, $nb=y'$, B$b=z'$: then (becauſe B*b* may be conſidered as a little right Line) by 47 *E*. 1. B$b = \overline{\overline{Bn}|^2 + \overline{nb}|^2}|^{\frac{1}{2}}$, *i.e.* $z' = \overline{x'^2 + y'^2}|^{\frac{1}{2}}$ or (*Art.* 3.) $\dot{z} = \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$. Put the Ordinate A B (*Fig.* 2.) $= y$, Curve A B $= z$; B$n = x'$, $nb = y'$, B$b = z'$: Then (becauſe B *b* may be conſidered as an indefinitely ſmall right Line, and B*n* as a little right Line perpendicular to A *b*) as before $z' = \overline{x'^2 + y'^2}|^{\frac{1}{2}}$ or $\dot{z} = \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$ and this is a general Expreſſion for the Fluxion of any curve Line whatſoever. Now by help of the Equation of the given Curve we may find the value of $\dot{x}_2$ in Terms of $\dot{y}^2$, or of $\dot{y}^2$ in Terms of $\dot{x}^2$; by which $\dot{x}^2$ or $\dot{y}^2$ in this general Expreſſion may be exterminated; and then if we find the Fluent of this we ſhall have z, or a definitive Expreſſion for the Length of the Curve Line required.

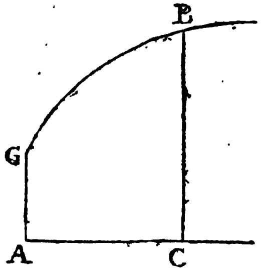
EXAMPLE I.

103. *To find the Length of the Curve* GB$=$ z, *whoſe Equation (putting the given Line* AG $= a$, AC$= x$, CB$= y$,) *is* $2 \times \overline{a^2 + x^2}|^{\frac{3}{2}} = 3a^2y$.

The Fluxion of this Equation (*Art.* 13.) is $3 \times \overline{a^2 + x^2}|^{\frac{1}{2}} \times 2x\dot{x} = 3a^2\dot{y}$, therefore $\dot{y} = \frac{2x\dot{x}}{a^2}\overline{a^2 + x^2}|^{\frac{1}{2}}$ and $\dot{y}^2 = \frac{4x^2\dot{x}^2}{a^4}\overline{a^2 + x^2} =$

$4a^2x$

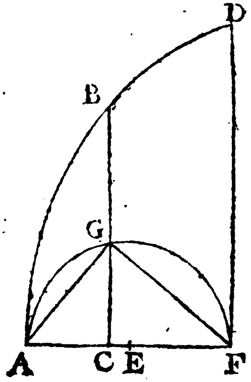
$\frac{4a^2x^2\dot{x}^2+4x^4\dot{x}^2}{a^4}$ which ſubſtituted for $\dot{y}^2$ in the general fluxional Expreſſion gives $\dot{z}=\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$ $=\overline{\frac{a^4\dot{x}^2+4a^2x^2\dot{x}^2+4x^4\dot{x}^2}{a^4}}|^{\frac{1}{2}}=\frac{a^2\dot{x}+2x^2\dot{x}}{a^2}=$the



Fluxion of the Curve G B, whoſe Fluent, by *Art.* 99. is $z=\frac{a^2x+\frac{2}{3}x^3}{a^2}$ or $x+\frac{2x^3}{3a^2}$ = the Length of the Curve G B required.

EXAMPLE II.

104. *To find the Length of the common Cyloid.*



PUT E A or E F, the Radius of the generating Circle $=a$, Abſciſs A C$=x$, Ordinate C B$=y$, C G$=s$, and A B$=z$, then (as may be found in *Art.* 45.) $\dot{y}=\frac{2a-x}{s}\dot{x}$ and therefore $\dot{y}^2=\frac{\overline{2a-x}|^2}{s^2}\dot{x}^2$, which ſubſtituted for $\dot{y}^2$ gives the general Expreſſion for the Fluxion of the Curve (*Art.* 102.) $\dot{z}=\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}=\overline{\dot{x}^2+\frac{\overline{2a-x}|^2}{s^2}\dot{x}^2}|^{\frac{1}{2}}=\frac{\overline{s^2\dot{x}^2+4a^2\dot{x}^2-4ax\dot{x}^2+x^2\dot{x}^2}|^{\frac{1}{2}}}{s^2}$ *i. e.* (becauſe by the Property of Circles $\overline{GC}|^2=AC\times CF$ or $s^2=2ax-x^2$) $\dot{z}=\overline{\frac{4a^2\dot{x}^2-2ax\dot{x}^2}{2ax-x^2}}|^{\frac{1}{2}}=\overline{\frac{2a\dot{x}^2}{x}}|^{\frac{1}{2}}=\overline{2a}|^{\frac{1}{2}}\times x^{-\frac{1}{2}}\times\dot{x}$, and the Fluent of this, by *Art.* 99. is $z=\overline{2a}|^{\frac{1}{2}}\times 2x^{\frac{1}{2}}=2\times\overline{2ax}|^{\frac{1}{2}}=$ twice the Chord A G (for the Triangles FAG and GAC being Similar, FA : AG : : GA : AC, or $\overline{FA\times AC}|^{\frac{1}{2}}=$ AG, *i. e.* $\overline{2ax}|^{\frac{1}{2}}=$AG.) Whence the Length of the Semi-Cycloid A D, is equal to twice the Diameter A F of its generating Circle.

CHAP. XI.

Of finding the Areas of Curve-lined Spaces.

105. IN Curves AB referred to an Axis (*Fig.* 1. laſt Chap.) let bc be conceived indefinitely near and parallel to the perpendicular Ordinate BC, and Bn equal and parallel to Cc; then, becauſe bn bears no aſſignable Ratio to BC, bc. may be taken as equal to BC or nc, And the Trapezium BCcb as equal to the Parallelogram BCcn; But BCcb is the Moment or Increment of the curvilineal Space ABC *i. e.* if we put $AC = x$, $CB = y$ $Cc = x'$, the Moment or Increment of the Space ABC is $= yx'$, and therefore its Fluxion (*Art* 3.) is $= y\dot{x}$.

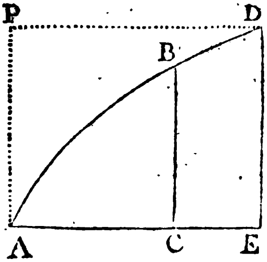
106. In like manner in Spirals or Curves referr'd to a fixt or central Point, let bA be conceiv'd indifinitely near to BA (*Fig.* 2. laſt Chap.) and Bn perpendicular to bA; then, bn having leſs than any aſſignable Ratio to nA, BAn may be conſidered as equal to BAb the Moment or Increment of the curvilineal Space BDA, *i. e.* if we put $AB = y$ and $Bn = x'$; the Moment or Increment of the Space ADB will be $= \frac{1}{2}yx'$; or its Fluxion (that is, the Ve-

locity with which it flows at the Point B) = $\frac{1}{2}y\dot{x}$.

107. Wherefore, when the Curve is referr'd to an Axis, find the Value of y in Terms of x by the help of the Equation of the given Curve, which multiply by $\dot{x}$; then the Fluent of this fluxional Expreſſion being found, will give the Area of the curvilineal Space B A C B requir'd (*ſee Fig.* 1. laſt Chap.) And when the Curve is referred to a central Point A, find the Value of $\dot{x}$ in Terms of $\dot{y}$, which may be done by help of the Equation of the given Curve, then multiply this Value of $\dot{x}$ by $\frac{1}{2}y$ and find the Fluent, and it will give the Area of the Space B D A B required (*ſee Fig.* 2. laſt Chap.)

EXAMPLE I.

108. *To find the Area of the common or apollonian parabolic Space* A D E A; *where* A E *is given* $= a$, *and* E D $= b$,



PUT AC$=x$, CB$=y$, and the Parameter $=p$; then by the Nature of the Curve $px=y^2$ or $y=\overline{px}|^{\frac{1}{2}}=p^{\frac{1}{2}}x^{\frac{1}{2}}$, which drawn into $\dot{x}$ is $y\dot{x}= p^{\frac{1}{2}}x^{\frac{1}{2}}$, and the Fluent of this (*Art.* 99.) is $\frac{2}{3}p^{\frac{1}{2}}x^{\frac{3}{2}}$ *i. e.* $\frac{2}{3}\overline{px^3}|^{\frac{1}{2}}$, or (by writing y^2 for px,) $\frac{2}{3}\overline{x^2y^2}|^{\frac{1}{2}}$ or $\frac{2}{3}xy$ which (*Art.* 107.) is $=$ the Area of the indeterminate Space ABCA; therefore, by ſubſtituting a for x and b for y, we have $\frac{2}{3}ab$ $=$ the Area ſought.

COROLLARY.

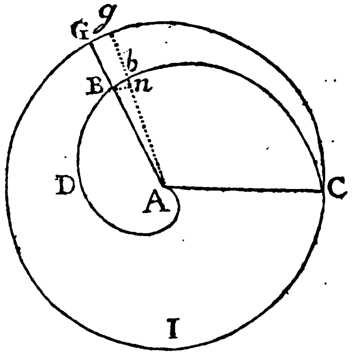
The Area of every common or apollonian parabolic Space ADEA, is always equal to two-third Parts of its circumſcribing Parallelogram APDE.

EXAMPLE II.

109. *To find the Area of the ſpiral Space* ADBCA *of* Archimedes.

PUT the Circumference of the generating Circle CIGC$=a$, and its Radius AC or AG $=b$; Ordinate AB$=y$, Arch CIG$=z$; let Ag be ſuppos'd indefinitely near to AG, and with the Ordinate AB as a Radius the little circular Arch Bn be deſcribed, *i. e.* let B$n=\dot{x}$, $nb=\dot{y}$ and G$g=\dot{z}$. Now, by the Nature of the Generation of the Curve, $b : a :: y$

$: z$,



$: z$, *i. e.* $z = \frac{ay}{b}$ in Fluxions $\dot{z} = \frac{a\dot{y}}{b}$, but by the ſimilar Sectors or Triangles AB*n* and AG*g*, $y : x' :: b : z'$, *i. e.* $z' = \frac{bx'}{y}$ or $\dot{z} = \frac{b\dot{x}}{y} \because \frac{b\dot{x}}{y} = \frac{a\dot{y}}{b}$ and $\dot{x} = \frac{ay\dot{y}}{b^2}$ which multiplied by $\frac{1}{2}y$ gives $\frac{1}{2}y\dot{x} = \frac{ay^2\dot{y}}{2b^2}$ whoſe Fluent (*Art.* 99.) is $\frac{ay^3}{6b^2}$, and (*Art.* 107.) = the Area of the Space ADBA, wherefore by ſubſtituting b for y, we have $\frac{1}{6}ab$ = the ſpiral Space required.

COROLLARY.

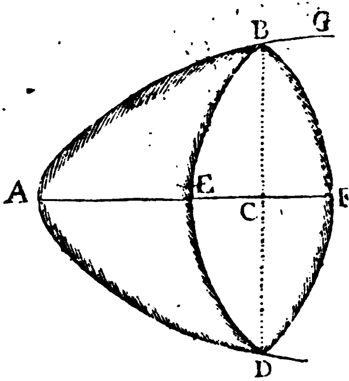
The Area of the ſpiral Space of *Archimedes*, is = one third Part of the generating Circle.

CHAP.

CHAP. XII.

Of finding the convex Superficies of Solids.

110. LET the Solid ACBEDF be conceived to be generated by the ſuperficial Fig. ACB revolving above AC as an

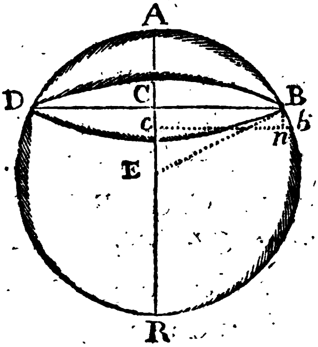


Axis; then the Velocity with which its convex Superficies flow, will be equal to the Velocity with which the Curve AG in its Deſcription flowed at the Point B, drawn into the Periphery of the Circle BEDFB deſcribed by the Radius CB, that is, the Fluxion of the convex Superficies of the Solid, is equal to the Fluxion of the Curve at the Point B drawn into the Periphery of a Circle whoſe Radius is CB.

111. Hence, if we put $AC=x$, $CB=y$, Curve $AB=z$, and $c=2\times 3.1416=$ the Circumference of a Circle whoſe Radius is 1, then cy will be equal to the Circumference of the Circle deſcribed by CB, and (*Art.* 102.) the Fluxion of the Curve $\dot{z}=\overline{\dot{x}^2+\dot{y}^2}\big|^{\frac{1}{2}}$; and the general Expreſſion for the Fluxion of the convex Superficies of any Solid will be $=cy\dot{z}$ or $cy\times\overline{\dot{x}^2+\dot{y}^2}\big|^{\frac{1}{2}}$; out of which, by help of the Equation of the given Curve AB, $\dot{x}^2$ or $\dot{y}^2$, &c. may be exterminated, and then the Fluent is found; which will give the Area of the convex Superficies required.

EXAMPLE I.

112. *To find the Superficies of a Sphere, or the convex Superficies of any Segment of it.*



PUT Radius EA or $EB=a$, $AC=x$, $CB=y$,

$AB = z$: Let $Bn = Cc$ expreſs the very firſt Moment of the Increaſe of AC, nb of CB, Bb of AB, *i.e.* let $Bn = x'$, $nb = y'$, and $Bb = z'$, then Bb being conſidered as a little right Line coinciding with a Tangent to the Point B, the Triangles ECB and bnB will be alike (for $\angle CBn = \angle EBb =$ a right Angle, therefore, $\angle EBn$ being common, $\angle CBE = \angle nBb$; and the Angles at C and n being right, the Angles CEB and Bbn muſt be likewiſe equal, *ergo*, &c.) Wherefore $EB : BC :: bB : Bn$, *i.e.* $a : y :: z' : x'$ $\therefore z' = \frac{ax'}{y}$ or (*Art.* 3.) $\dot{z} = \frac{a\dot{x}}{y}$, which ſubſtituted for $\dot{z}$, gives the general Expreſſion for the Fluxion of the convex Superficies $cy\dot{z}$ (*Art.* 111.) $= cy \times \frac{a\dot{x}}{y} = ca\dot{x}$, whoſe Fluent is $cax =$ the convex Superficies of the Segment ACBD: And if for x be ſubſtituted $2a$, we have $2ca^2 =$ the Superficies of the whole Sphere.

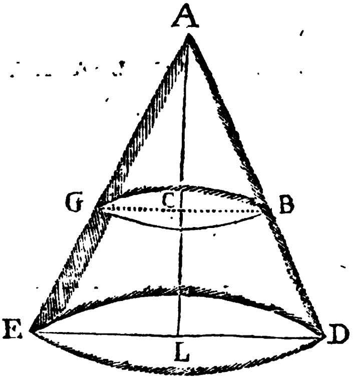
COROLLARIES.

1. THE convex Superficies of any Segment of a Sphere, is equal to the Periphery of a great Circle of that Sphere multiplied into the Altitude of the Segment.

2. The whole Surface or Superficies of any Sphere, is equal to the Periphery of its greateſt Circle multiplied into its Diameter.

EXAMPLE II.

113. *To find the concave Superficies of the right Cone* ADE, *whoſe Altitude* AL *is given* $= a$, *and Baſe diameter* DE $= b$.



PUT AC $= x$, CB $= y$. By Sim. $\triangle$s AL : LD :: AC : CB, *i.e.* $a : \frac{1}{2}b :: x : y \because x = \frac{2ay}{b}$ which put into Fluxions is $\dot{x} = \frac{2a\dot{y}}{b} \because \dot{x}^2 = \frac{4a^2\dot{y}^2}{b^2}$ which ſubſtituted for $\dot{x}^2$ in the general Expreſſion for the Fluxion of the convex Superficies, gives $cy \times \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$ (*Art.* 111.) $= cy \times \overline{\frac{4a^2\dot{y}^2}{b^2} + \dot{y}^2}|^{\frac{1}{2}} = \frac{c}{b} \times \overline{4a^2 + b^2}|^{\frac{1}{2}}\, y\dot{y}$, whoſe

Fluent is $\frac{c}{2b} \times \overline{4a^2+b^2}|^{\frac{1}{2}} y^2 = \frac{cy^2}{b} \times \overline{a^2 \times \frac{1}{4}b^2}|^{\frac{1}{2}}$ $=$ (becauſe $AD = \overline{\overline{AL}|^2 + \overline{LD}|^2}|^{\frac{1}{2}} = \overline{a^2+\frac{1}{4}b^2}|^{\frac{1}{2}}$) $\frac{cy^2}{b} \times AD =$ the convex Superficies of the Cone ABG generated by the plain Figure ACB; and by ſubſtituting $\frac{1}{2}b$ for y we have $\frac{1}{4}cb \times DA =$ the convex Superficies of the Cone ADE.

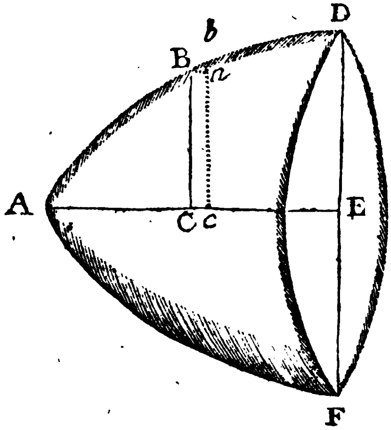
COROLLARY.

The convex Superficies of any right Cone, is equal to half the Circumference of its Baſe multiplied into its ſlant Height.

CHAP. XIII.

Of finding the Contents of Solids.

114. LET bc be conceived indefinitely near and parallel to the variable Ordinate BC; and Bn equal and parallel to Cc, the Increment of the variable Abſciſs AC; then, becauſe the little Parallelogram BCcn is expreſſive of the Increment of the plain Fig. ACB (*Art.* 105.) therefore, if a Solid AEDF be conceived to be generated by the Revoluti- of the curvilineal plane Fig. AED round the Axis AE, the indefinitely little Cylinder generated



rated by the ſaid little Parallelogram will expreſs the Moment or Increment of the Solid at B; and, becauſe this Moment or Increment is equal to the Area of the Circle deſcribed by the Ordinate C B drawn into the Increment of the Abſciſs A C; therefore the Fluxion of the Solid at B, is equal to the Area of a Circle whoſe Radius is C B drawn into the Fluxion of the Abſciſs A C; which, if we put $AC=x$, $CB=y$, and $c=3.1416=$ the Area of a Circle whoſe Radius is 1, is $=cy^2\dot{x}$: And this is a general Expreſſion for the Fluxion of any Solid A E D F at any Point B whatſoever; out of which, by help of the Equation of the given Curve A B, either $\dot{x}$ or y^2 may be exterminated, and then the Fluent found, which will give the Content of the Solid generated by ACB and then if for x or y we ſubſtitute the Value of AE or ED we ſhall have the Content of the Solid AEDF required, as in the following Examples.

EXAMPLE I.

115. *To find the Content of a Sphere or any Segment of it.*

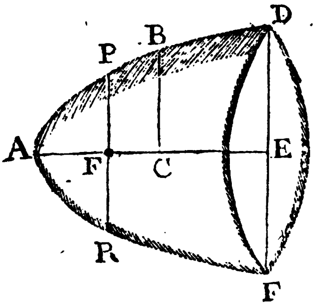
PUT $AC = x$, $CB = y$, Diameter $AR = a$, (*ſee Fig.* 2. in the laſt Chap.) then by the Property of Circles $AC \times CR = \overline{CB}|^2$, *i. e.* $ax - x^2 = y^2$. Now by ſubſtituting $dx - x^2$ for y^2 in the general Expreſſion for the Fluxion of the Solid Content we have $cy^2\dot{x}$ (*Art.* 114.) $= c\dot{x} \times \overline{ax - x^2} = cax\dot{x} - cx^2\dot{x}$, whoſe Fluent is $\frac{cax^2}{2} - \frac{cx^3}{3} = \frac{3cax^2 - 2cx^3}{6} =$ the Content of the Segment ACBD : and if a be ſubſtituted for x we ſhall have $\frac{3ca^3 - 2ca^3}{6} = \frac{1}{6}ca^3 =$ the Content of the whole Sphere ABRD. Hence, becauſe four times the Area of a great Circle of the Sphere is $= ca^2$, and the Content of the Cylinder circumſcribing the Sphere is $= \frac{1}{4}ca^3$, we have the following

COROLLARY.

The Content of any Sphere is equal to four Times the Area of its greateſt Circle multiplied into $\frac{1}{6}$th Part of its Axis; or equal to two-third Parts of its circumſcribing Cylinder.

EXAMPLE II.

116. *To find the Content of the Parabolic Conoid* AEDF *generated by the Apollonian or common Semi-Parabola* AED *revolving round the Axis* AE; *where* $AE=b$, $ED=d$.



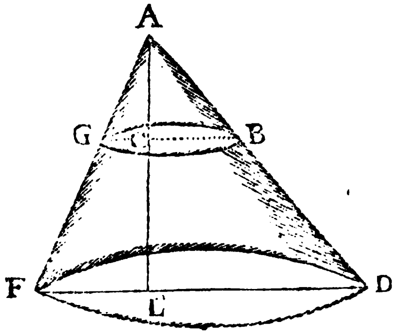
PUT the Parameter PR, which paſſes thro' the Focus F of the Parabola $=a$; Abſciſs $AC=x$; and Ordinate $CB=y$; then by the Nature of the Parabola $ax=y^2$. Now by ſubſtituting ax for y^2 we ſhall have the general Expreſſion for the Fluxion of the Solid $cy^2\dot{x}$ (*Art.* 114.) $=cax\dot{x}$, whoſe Fluent is $\frac{1}{2}cax^2=$ (by writing y for ax) $\frac{1}{2}cxy^2 =$ the Content of the Parabolic Conoid generated by the Semi-Parabola ACB, and by ſubſtituting b for x and d for y, we have $\frac{1}{2}cbd^2 =$ the Content of the Conoid required.

COROLLARY.

The Content of every common Parabolic Conoid is equal to $\frac{1}{2}$ of its circumſcribing Cylinder.

EXAMPLE III.

117. *To find the Content of any Cone whoſe Baſe is a Circle.*



PUT the given Altitude $AL=a$, Baſe-Diameter $DF=b$, let BG be parallel to DL and put $AC=x$, and $c=.7854$ then, by Sim. △s $AL : DF :: AC : BG$, *i. e.* $a : b :: x : \frac{bx}{a}$ $=BG$

$=$ B G therefore the Area of the Circle B G $=$ $\overline{\frac{bx}{a}}\Big|^{2} \times c = \frac{cb^2x^2}{a^2}$ which multiplied by $\dot{x}$ (*Art.* 114.) is $\frac{cb^2x^2\dot{x}}{a^2}$ $=$ the Fluxion of the Content of the Cone at B, whoſe Fluent is $\frac{cb^2x^3}{3a^2}$ $=$ the ſolid Content of the Cone A B G; and by ſubſtituting a for x we have $\frac{cb^2a^3}{3a^2} = \frac{1}{3}cb^2a^2 =$ the Content of the Cone A D F required.

COROLLARY.

The ſolid Content of any Cone whoſe Baſe is a Circle, is equal to the Area of its Baſe multiplied into $\frac{1}{3}$ of its perpendicular Altitude.

CHAP. XIV.

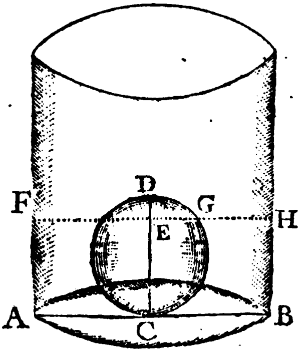
Some Miſcellaneous Queſtions with their Anſwers.

I.

THERE is a cylindrical Tube, whoſe Diameter is four Inches, in which is contained 18 cubic Inches of Water: Now ſuppoſing a heavy Sphere, whoſe Axis is two Inches, to be thrown into this Tube; 'tis required to find what Part of the ſaid Sphere will be immerſed in the Water, and the Altitude of the Segment above the Surface of it.

SOLUTION.

PUT DC the Diameter of the Sphere $=2=a$; and EC that Part of it which is $=$ the Altitude of the Water after the Sphere is thrown

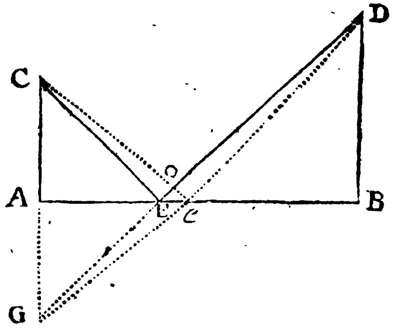


into it $=x$; AB the Diameter of the Tube $=4=b$; the Content of the Water in the Tube $=18=c$; and $.7854=f$. Then $ED=a-x$, and by the Property of Circles $DE\times EC=\overline{EG}|^2$, *i. e.* $ax-x^2=$ the □ of the Radius of the Plane of the Section of the Sphere; therefore the Area of this Plane is $=4fax-4fx^2$ which multiplied into $\dot{x}$ is $4fax\dot{x}-4fx^2\dot{x}$ $=$ the Fluxion of the Segment of the Sphere under the Water, whoſe Fluent is $2fax^2-\frac{4}{3}fx^3+c=$ the Content of the Cylinder ABHF

fb^2x, by Tranſpoſition $\frac{4}{3}fx^3 - 2fax^2 + fb^2x = c$ and by dividing both Sides of this Equation by $\frac{4}{3}f$ we have $x^3 - \frac{3}{2}ax^2 + \frac{3}{4}b^2x = \frac{3c}{4f} = 17.18869.$ whence x is found $= 1.76 = BE$, and $a - x = 0.24 = ED$ the Altitude of the Segment above the Water. *Q. E. I.*

II.

Given A B, A C, *and* B D. *Quære the Ratio of the Angles* A E C *and* B E D, *when* C E + E D *is a* Minimum?



SOLUTION.

It is evident that the Fluxions of the Lines CE and DE are Negative to each other, and becauſe the Sum of theſe Fluxions is = o, therefore they are equal to each other, or, becauſe the Increment may be taken for the Fluxion, if e be conceived indefinitely near to E, then

$oe=oE$ and conſeq. $\angle oEe=\angle oeE$, *i. e.* the Angles DEB and CEA are in the Ratio of Equality. *Q. E. I.*

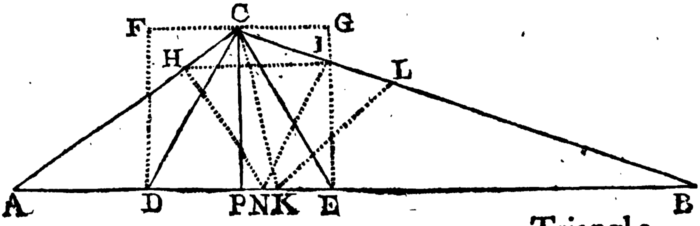
Or, by a plain geometrical Way of Reaſoning, thus. —— Produce CA to G, making $AG=AC$; then will $GE=EC$, and $\angle AEG = \angle AEC$: But the Sum of GE and ED is the leaſt poſſible when they are in the ſame Direction, *i. e.* when $\angle AEG=\angle BED$, as is evident on the leaſt Conſideration; therefore, when $CE+ED$ is a *Minimum*, the Angles AEC and BED are equal.

III.

Quære the greateſt equilateral Triangle that can be inſcribed in a ſcalene Triangle whoſe Sides are 4, 7, *and* 10?

SOLUTION.

As the Angles of an equilateral Triangle are 60° each, it is plain that the extreme Point of either of its Angles cannot fall in with the extreme Point of either of the Angles of the given



Triangle except the largeſt, *viz.* ths ∠ACB. Let CDE then be the greateſt, and put its Side $CD = s$, let drop the Perpendicular CP; then by 13 *E.* 2. $AP = \frac{\overline{AC}^2 + \overline{AB}^2 - \overline{BC}^2}{2AB}$ $= 3.35$; and by 47 *E.* 1. $\overline{AC}^2 - \overline{AP}^2 = \overline{PC}^2$ $= \overline{CD}^2 - \overline{DP}^2$, *i. e.* $\overline{AC}^2 - \overline{AP}^2 = s^2 - \frac{1}{4}s^2$ or $3.7775 = \frac{3}{4}s^2$, whence $s = 2.244$. *Q. E. I.*

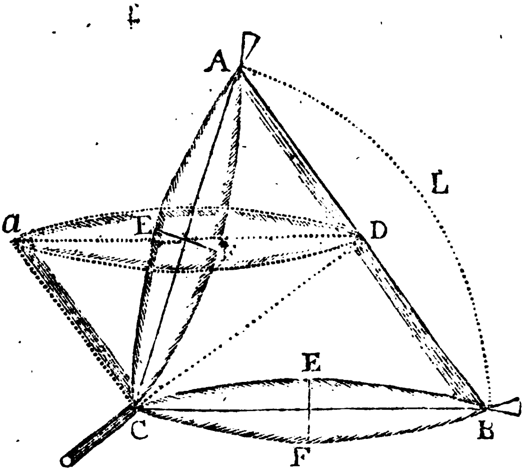
N. B. In Order to prove that the Triangle CED is the greateſt Equilateral that can be inſcribed, from the Points D and E erect the Perpendiculars DF and EG equal to PC, and draw the Line FG; then will the Triangle CDE be $= \frac{1}{2}$ the Parallelogram DFGE. Let another Triangle be drawn, which, if it be *within* the Parallelogram as HIN, 'tis plain it cannot be $= \frac{1}{2}$ the Parallelogram, becauſe it hath no Side coincident therewith; and if it be *without* and have one Side coincident with one Side of the given Triangle as CLK, neither then can it be $= \frac{1}{2}$ the Parallelogram; for if ſo, CK would be $= CE$, and as $\overline{CE}^2 - \overline{EP}^2$ $= \overline{CN}^2 - \overline{KP}^2$, then ſubtracting $\overline{CK}^2 = \overline{CE}^2$ from the Equation, $\overline{PK}^2 = \overline{PE}^2$ or $PK = PE$ which is impoſſible, therefore CDE is the greateſt Equilateral that can be inſcribed. *Q. E. D.*

Suppoſe

IV.

Suppoſe the Sides of a Pair of Bellows to be two equal Circles: Required the Inclination of Planes when they contain the greateſt Quantity of Air.

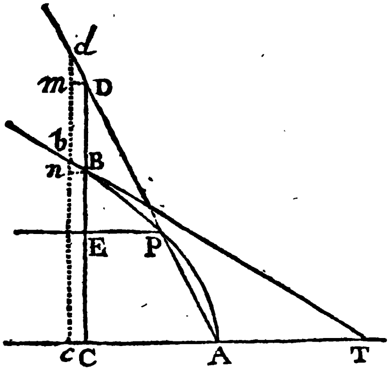
SOLUTION.



Let AC and BC be the two Sides, and C the Center on which they turn. Now it is plain that if AC be moved up from BC the Point A will deſcribe an Arch of a Circle BLA, and the Chord of that Arch AB will be the Side of the Solid formed by the Bellows ſo extended: Let us ſuppoſe this Solid cut by a plane Perpendicular to AB; then becauſe the Diameter AC is contracted into the Coſine DC while the normal Diameter EF continues always in a parallel Situation, and therefore is not at all diminiſhed; it follows that the Plane of that Section will be an Ellipſis: And as the Parts ACD and BCD are equal, if we conceive the upper Part ACD to be moved round into the Situation *a*CD, it will form an oblique Cylinder *a*CBD, and conſequently the Content will be greateſt when the Altitude is ſo, *i. e.* when the Sides of the Bellows are normal to each other. *Q. E. I.*

V.

Quære the Nature of the Curve APB, *ſuppoſing* CE *the Diſtance of the parallel Lines*



AC *and* PE, *as alſo the Angle* CAD, *to be given, and* $\overline{CE}|^{m} \times \overline{CD}|^{n} = \overline{CB}|^{m+n}$?

SOLUTION.

Put $CE=a$, $CD=z$, Abſciſs $AC=x$, Ordinate $CB=y$. Let dc be conceived indefinitely near and parallel to DC; and Dm, Bn, equal and parallel to Cc; and put $nb=y'$, $md=z'$; and ſuppoſe BT a Tangent to the Curve at the Point B. Then $DC : CA :: dm : md$, *i.e.* $z : x :: z' : \frac{xz'}{z} = Dm$ or Bn: And $bn : nB :: BC : CT$, *i.e.* $y' : \frac{xz'}{z} :: y : \frac{xyz'}{zy'} = CT$, or, ſubſtituting the Fluxion for the Increments, $\frac{xy\dot{z}}{z\dot{y}} = CT$. But by the Queſtion $a^m z^n = y^{m+n}$ which in Fluxions is $na^m z^{n-1}\dot{z} = \overline{m+n}\, y^{m+n-1}\dot{y}$, $\because \dot{z} = \frac{\overline{m+n}\, y^{m+n-1}\dot{y}}{na^m z^{n-1}}$, and this ſubſtituted for $\dot{z}$ in $\frac{xy\dot{z}}{z\dot{y}}$ gives $CT =$

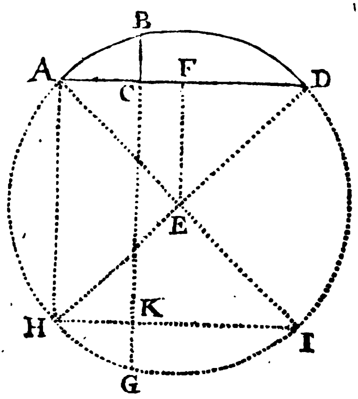
$$\frac{x.\overline{m+n}\, y^{m+n}}{na^m z^n} = (\text{becauſe } \frac{y^{m+n}}{a^m z^n} = 1,)\ \frac{m+n}{n}x.$$

Now this is an univerſal Expreſſion for the Subtangent of all poſſible Parabolas; therefore when $m=n=1$, the ſubtangent will be $2x$, and the Curve the common or Apollonian Parabola. *Q. E. I.*

Note, Tho' this univerſal Expreſſion for the Subtangent, ſeems to differ from that found in *Art.* 39. yet it will be found to expreſs the very ſame thing, if we conſider that the Parameter drawn into x, in the Equation of the Curve there, is equal to $a^{\frac{m}{n}}z$ here, for then, m there, will be equal to $\frac{m+n}{n}$ here.

VI.

In the Curve ABD, *whoſe Property (putting* $AC=x$, $CB=y$, *and* $a=$ *a given Line,) is* $ax-ay=x^2+y^2$; *required the Radius of Curvature for any Point; and a geometrical Conſtruction to illuſtrate and confirm the Work.*



SOLUTION.

The Fluxion of the given Equation of the

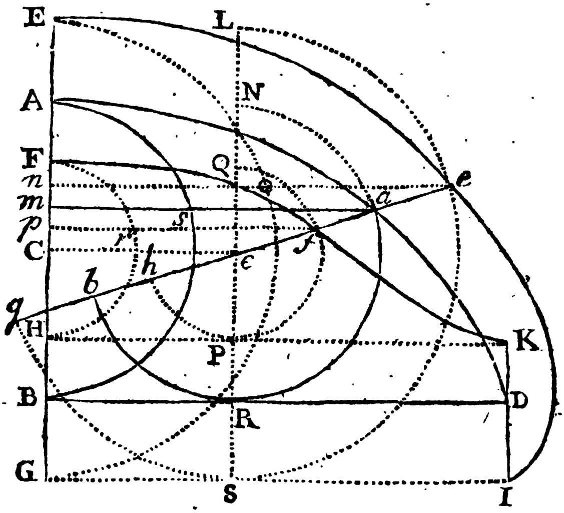
Curve, making $\dot{x}=1$, is $a-a\dot{y}=2x+2y\dot{y}$, and this fluxional Equation put into Fluxions again, the Fluxion of $\dot{y}$ being Negative, is $a\ddot{y}=2+2\dot{y}^2-2y\ddot{y}$. Hence we have $\dot{y}=\frac{a-2x}{a+2y}$, $\dot{y}^2=\frac{a^2-4ax+4x^2}{a^2+4ay+4y^2}$, and $\ddot{y}=\frac{2+2\dot{y}^2}{a+2y}$, *i.e.* by ſubſtituting for $\dot{y}^2$ its equal, $\ddot{y}=\frac{4a^2+8ay+8y^2-8ax+8x^2}{\overline{a+2y}|^3}$ or becauſe by the Queſt. $8x^2+8y^2=8ax-8ay$, $\ddot{y}=\frac{4a^2}{\overline{a+2y}|^3}$. Now by ſubſtituting for $\dot{y}^2$ and $\ddot{y}$, theſe their Values, in the general Expreſſion for the Radius of Curvature, which was found *Art.* 64. $=\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{\ddot{y}}$, when $\dot{x}=1$, and the Fluxion of $\dot{y}$ Negative; we have $\overline{\frac{2a+4ay+4y^2-4ax+4x^2}{\overline{a+2y}|^2}}\Big|^{\frac{3}{2}}$ $\times\frac{\overline{a+2y}|^3}{4a^2}=$ (becauſe $4x_2+4y^2=4ay-4ax$,) $\overline{\frac{2a^2}{\overline{a+2y}|^2}}\Big|^{\frac{3}{2}}\times\frac{\overline{a+2y}|^3}{4a^2}=\frac{\overline{2a^2}|^{\frac{3}{2}}}{4a^2}=\frac{\overline{8}|^{\frac{1}{2}}a}{4}=\frac{a}{\overline{2}|^{\frac{1}{2}}}=$ the Radius of Curvature for any Point required, which being a fixt or invariable Quantity, proves the Curve to be a Circle.

Now, if the Radius of a Circle be $\frac{a}{\overline{2}|^{\frac{1}{2}}}$, 'tis evident that a is the Side of the inſcribed Square, or the Chord of 90° as AD; Therefore, if AD be biſected in F, and the Perpendicular FE drawn $= AF = \frac{1}{2}a$; then the Point E will fall in the Center of the Circle, and conſequently $AE\ (= \overline{EF|^2 + FA|^2}|^{\frac{1}{2}}) = \frac{a}{2|^{\frac{1}{2}}}$. And that x in the given Equation muſt flow in this Line, may be thus demonſtrated: To any Point C draw BC perpendicular, and let it be produced till it meet the Circle's Periphery in G, and let HI be drawn parallel and equal to AD, then 'tis evident that $CK = AH = AD$, by Conſtruction; and $KG = CB$, becauſe $AC = HK$ and $AB = HG$; therefore $CG = AD + CB$: But by a Property of Circles $AC \times CD = BC \times CG$, *i. e.* (putting $AD = a$, $AC = x$, $CB = y$,) $x \times \overline{a - x} = y \times \overline{a + y}$ or $ax - x^2 = ay + y^2$.

From the Conſtruction here given, it appears that the Queſtion may be infinitely diverſified, ſo as to be adapted to any regular Polygon of an even Number of Sides that can be inſcribed in a Circle.——We ſee here alſo a Demonſtration of the Juſtneſs of the fluxional Calculus, as made uſe of above.

VII.

If a Semicircle A B, be rolled along upon a right Line B D, untill it meaſure out a Line equal to its Circumference; then, the Curve



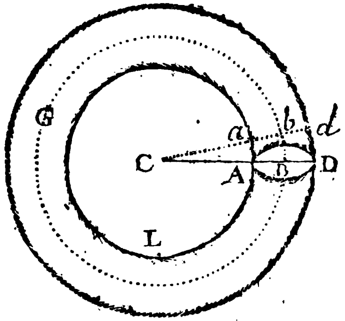
A D, deſcrib'd by the Point A, being the common Semicycloid; I ſay the Curve E I, deſcribed by the Point E taken *without* the Circle, will be the Contracted or Curtate Semi-Cycloid; and the Curve F K, deſcribed by the Point F taken *within* the Circle, will be the Inflected Semicycloid. Quære the Demonſtration?

DEMONSTRATION.

As all the Cycloids, properly ſpeaking, are generated by the Revolution of one and the ſame Circle A B; ſo therefore their Baſes will be all equal to B D, *viz.* = the Semicircle AB. Let the Generant move on till it come in the Poſition ab thro' the deſcribing Points a, e, f, draw the Ordinates ma, ne, pf, and draw LS parallel and equal to E G; then, becauſe the correſponding Ordinates in the Circle are reſpectively equal, it is evident that the Diſtances from the Extremities of theſe Ordinates to the correſpondent Points in the Curves will be all equal to the Diſtance moved over by the common Center C, that is, $sa = oe = rf$, and this alſo is = B R = Arch bR by the Generation. Now the Sectors Lce and Scg, or Qcf and Pcb are equal and Similar, and Similar to to the equal Sectors Nca and Rcb. Therefore, $cg : cb ::$ gS : bR, or (if we put the Semicircle $eg = a$, Semicircle ab or Baſe B D $= b$,) as $a : b ::$ Arch gS : Arch bR :: Eo : (B R) oe, *i. e.* Semicircle E G : G I :: Eo : oe, wherefore EeI is the Curtate Semicycloid. And $cb : cb :$ bP : bR or (if we put Semicircle $fb = a$, and Semicircle ab or Baſe B D $= b$,) as $a : b ::$ bP : bR :: Fr : (B R) rf, *i. e.* Semicircle F H : H K :: Fr ; rf, wherefore the Curve FfK is the inflected Semicycloid. *Q. E. D.*

VIII.

Quære the Content of the cylindrical Ring ABDG? *the Radius* CA *of the inner Circumference being given* $=a$; *and* AD *the Diameter of the Ring, or generating Circle*, $=b$.



Put any Arch LA$=x$; .7154$=c$; ſuppoſe Cd indefinitely near to CD; and draw the concentric Circle BG, making AB$=$BD $=\frac{1}{2}b$: Then may Dd, Bb, Aa, be conſidered as indefinitely ſmall right Lines; and therefore the Moment of the Ring will be equal to the Area of the Circle ABD drawn into the Increment Bb. Now CA : Aa :: CB : Bb (*i. e.*) $a : x' :: a+\frac{1}{2}b : x'+\frac{bx'}{2a}=Bb$; and the Area of the Circle AD$=b^2c$; therefore the Moment of the Ring is $=b^2c\times\overline{x'+\frac{bx'}{2a}}=b^2cx'+\frac{b^3cx'}{2a}$ or its Fluxion$=b^2c\dot{x}+$

$b^3c\dot{x}$

$\frac{b^3c\dot{x}}{2a}$, and the Fluent of this is $b^2cx+\frac{b^3c\dot{x}}{2a}=\overline{1+\frac{b}{2a}}\times b^2cx=$ the Content of the Ring from L to A; and by ſubſtituting $8ac$ for x, we have the Content of the whole Ring $=\overline{1+\frac{b}{2a}}\times 8ab^2c^2=$ the Area of the generating Circle A D drawn into the Circumference of the prickt Circle B G. *Q. E. I.*

IX.

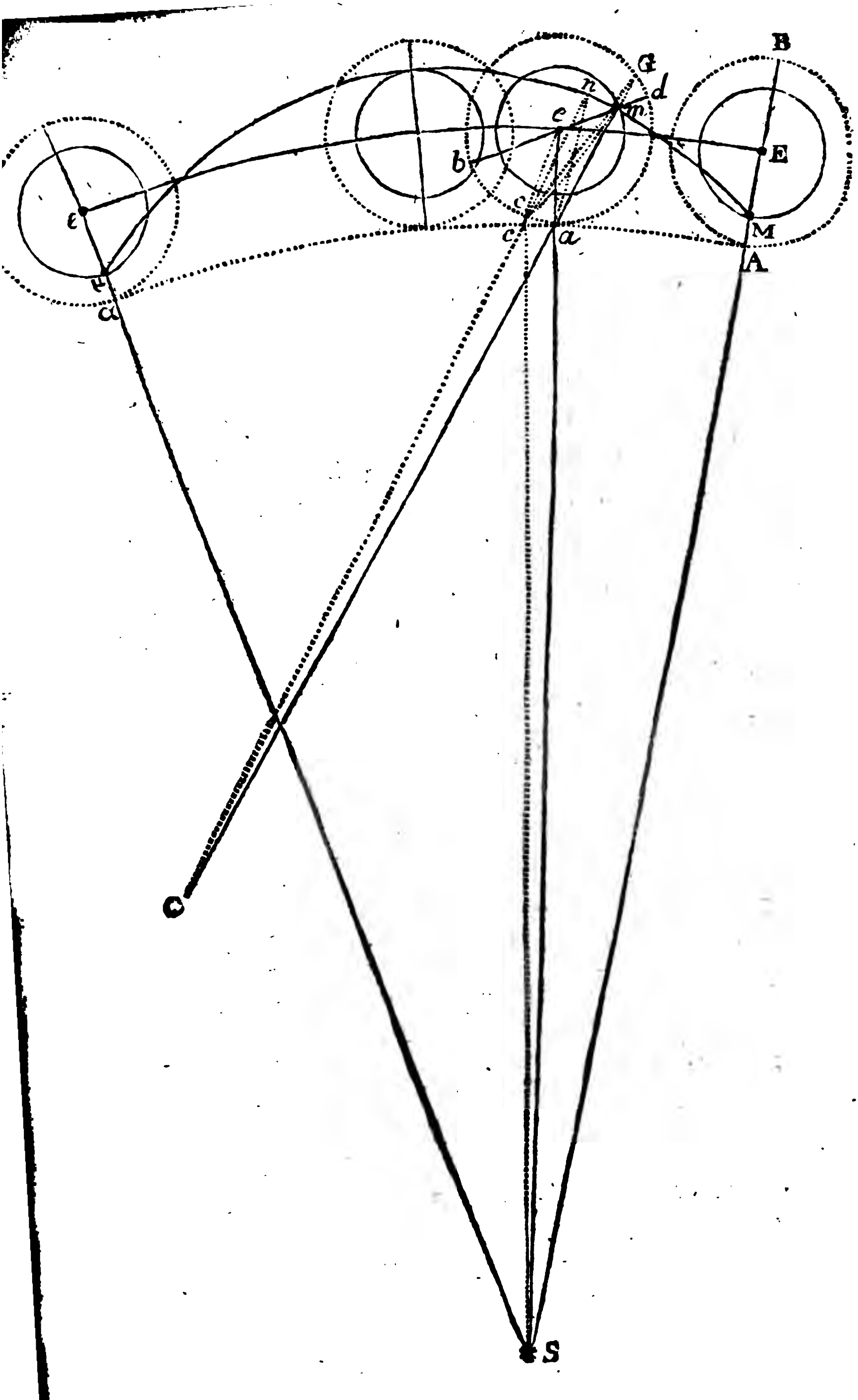
Suppoſe the Earth to revolve in a circular Orbit round the Sun as its Center; and the Moon to revolve round the Earth in the ſame Manner; as alſo that the Planes of their Orbits do coincide; and that the Diameters of the ſaid Orbits are as 340 *to* 1; *and laſtly, that the Moon performs* 13.368 *Revolutions to every ſingle Revolution of the Earth. Quære the Nature and Deſcription of the Curve generated by the Center of the Moon?*

[See the *Gentleman's Magazine* for *September* 1743, and the *Supplement* for that Year.

SOLUTION.

Let S repreſent the Sun; E the Earth; E *e*, an Arch of the Orbit of the Earth paſſed over by its Center in one Lunation of the Moon;

the Circumference of the Circle E A B = the concentric Arch Aa: Then (becauſe 13.368 —1 = 12.368 is = the Number of Lunations in a Year, or one Revolution of the Earth,) when the Moon is in Conjunction with the Sun, the Diſtance between the Sun and Moon, will be greater than the Diſtance or Radius S A. Now the Curve deſcribed by the Center of the Moon, is the ſame as that deſcribed by a Point M (E M being the Semidiameter of the Moon's Orbit) carried round by the Rotation of the Circle E A B on the Arch Aa; it is therefore of the cycloidal Kind, having a Point of contrary Flexure, if a Cycloid deſcribed by *any* Point within the generating Circle has a Point of Inflection as well upon a circular as upon a rectilinear Baſe. In Order to determine which, put S A or S$a = a$, E A or $ea = b$; E M or $em = c$, and $am = r$, and aG $= s$; and let mC be the Radius of Curvature at any Point m, and nC indefinitely near it; and let ac, ac, be the indefinitely ſmall contemporary Arches with mn, and conſequently the Triangles amc and anc equal in all reſpects, and the $\angle man = \angle cac =$ (becauſe the Angles eac and Sac added to either Side of the Equation makes it 2 right Angles,) $\angle aec + \angle a\mathrm{S}c$. Now as S$a$: ea :: $\angle aec$: $\angle a\mathrm{S}c$



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$\angle aSc$; and Sa (a): S$a+ae$ ($a+b$) :: $\angle aec$: $\angle aec+\angle aSc$, or $\angle man=\frac{a+b}{a}\angle aec$: Again, in any Triangle as Gmc, if the Angles mGc, mcG, and amc the Complement of the obtuſe Angle to 2 right Angles, be indefinitely ſmall, they are proportional to the oppoſite Sides mc, mG, and Gc, (*i. e.*) as Gc : mG :: $\angle amc$: $\angle mc$G, and G$c-m$G, or ma (r) : Gc (s) :: $\angle amc-\angle mc\text{G}=\angle a\text{G}c$ or $\frac{1}{2}\angle aec$: $\angle amc$ or $\angle anc=\frac{s}{2r}\angle aec$; and again $\angle m\text{C}n$, or $\angle man-\angle anc$ $(=\overline{\frac{a+b}{a}-\frac{s}{2r}}\angle aec)$: $\angle anc$ $(=\frac{s}{2r}\angle aec)$ :: an (r) : $a\text{C}=$ $\frac{ars}{2ar+2br-as}$ Conſequently $ma+a\text{C}=m\text{C}=$ $\frac{2ar^2+2br^2}{2ar+2br-as}=\frac{r^2}{r-\frac{as}{2a+2b}}=$ the Radius of Curvature at the Point m. Now as this Expreſſion for the Radius of Curvature muſt become Negative on the other Side of the Point of Inflection; r muſt be more than $\frac{as}{2a+2b}$ on one Side of the ſaid Point, and leſs on the other, and in that Point of Inflection $r=\frac{as}{2a+2b}$.

And therefore $Gm \times ma\ (=rs-r^2) = \frac{2abs^2+a^2s^2}{\overline{2a+2b}|^2}$ = (by the Property of Circles) $bm \times md = b^2-c^2$, and $s = \frac{\overline{2a+2b}\sqrt{a^2-c^2}}{2ab+b^2}$: Or to find r ſay $2ar+2br=as$, then $Gm \times md\ (= rs-r^2) = \frac{ar^2+2br^2}{a}$ and (becauſe $Gm \times md = bm \times md) = a^2-c^2$, therefore $r = \sqrt{\frac{ab^2-ac^2}{a+2b}}$, when the Point m falls in the Point of Inflection; but as ma (r) muſt always be greater than md, that is, as $\sqrt{\frac{ab^2-ac^2}{a+2b}}$ muſt always be greater than $b-c$, ſo therefore (em) c muſt be always greater than $\frac{b^2}{a+b}$ in order to a Point of Inflections taking Place in the Curve. Now, a, b, and c, being as 12.368, 1, and $\frac{13.368}{340}$ or 0.03911, therefore $\frac{b^2}{a+b}$ is greater than c; conſequently the Curve $Mm\mu$ generated by the Center of the Moon has not a Point of contrary Flexure, or is no where Convex towards the Sun. *Q. E. I.*

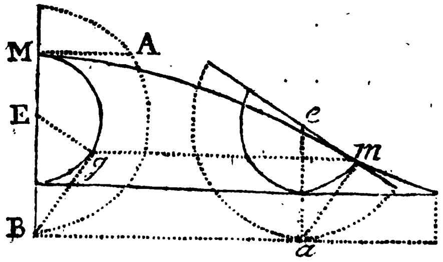
COROLLARIES.

1. When $r = s$, that is, when the Point m

coincides with d, the Radius of Curvature will be $= \frac{2ar^2+2br^2}{ar+2br}$ or $\frac{a+b}{a+2b}2r$, and $aC = \frac{a}{a+2b}r$, by Analogy $a+2b : a :: r : aC$, that is $SB : SA :: ma : aC$.

2. When a is infinite and $r=s$, that is, when the Baſe becomes a right Line, and the Point m coincides with d, or the Curve is the common Cycloid; the Radius of Curvature will be $=2r$; For then $2br^2$ and $2br$ will be infinitely little in compariſon of $2ar^2$ and ar, and therefore may be rejected.

3. When a is infinite, that is, when the Baſe degenerates into a right Line, or the Curve is the common Inflected or Interior Cycloid, as in



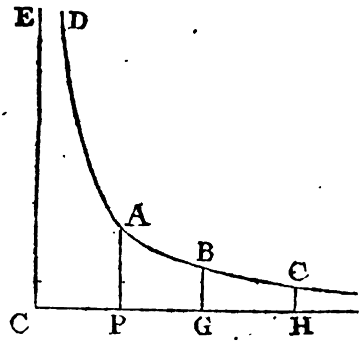
the annexed Fig. if the Tangent Bg be drawn, the Point of Inflection will be in the Line gm parallel to the Baſe; that is, if AM be perpendicular to MB, when the Point A comes to the Baſe, the Point M will be in the Point of Inflection. For when the Point m is in the Line gm, it is evident that am will be equal to Bg, and the Triangles BgE and ame, equal in every reſpect; and, becauſe the Tangent to a Circle is perpendicular to the Radius, the $\angle ame =$ a right Angle; wherefore, by 47 *E*. 1. $\overline{ae}|^2 - \overline{em}|^2 = \overline{ma}|^2$; that is, (if m be the Point of Inflection, and not otherwiſe,) $b^2 - c^2 = r^2 = \frac{ab^2 - ac^2}{a + 2b}$, *i. e.* $b^2 - c^2 = \overline{b^2 - c^2} \times \frac{a}{a + 2b}$ but $2b$ being infinitely little in reſpect of a, therefore $\frac{a}{a + 2b}$ is infinitely near to Equality with $\frac{a}{a}$, *i. e.* $= 1$; *ergo*, &c.

X.

The Fluxion of the Hyperbolic Logarithm of any Quantity, is equal to the Fluxion of that Quantity divided by the Quantity itſelf. Quære the Demonſtration? (See Art. 14.)

SOLUTION.

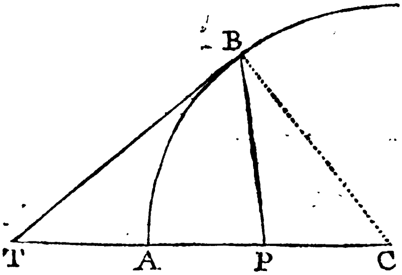
Let DABC be an Hyperbola, whoſe Aſymptotes are CE and CH, and its Parameter AP, which by a Property of the Curve is = PC. Now if AP or PC be made = 1, and GB or HC drawn parallel to PA; then the Space



Space PABG will be the hyperbolic Logarithm of CG; and the Space PACH the hyperbolic Logarithm of CH; or the Space GBCH the hyperbolic Logarithm of $\frac{CH}{CG}$: For by the Property of the Curve when CP, CG, CH, &c. are in geometrical Proportion continued, the Spaces PABG and GBCH, &c. will be equal; that is, the Spaces PABG, PACH, &c. will be in arithmetical Progreſſion. So that the Fluxion of the Space PABG is the Fluxion of the hyperbolic Logarithm of CG: Now the Fluxion of this Space, putting $PG+x$ and GB, perpendicular to $GC=y$, is $=\dot{x}y$ by *Art.* 105. but by the Nature of the Curve CG : CP :: PA : GB, *i. e.* $1+x : 1 :: 1 : y = \frac{1}{1+x}$ wherefore $\frac{\dot{x}}{1+x}=$ the Fluxion of the hyperbolic Logarithm of $1+x$. But the Fuxion of $1+x$ divided by $1+x$ is likewiſe $=\frac{\dot{x}}{1+x}$, or the Fluxion of the Space PABG, is equal to the Fluxion of C G divided by C G itſelf. *Q. E. D.*

XI.

Given the Point P *in the Radius* C A *of the given Circle* AB, *&c. Quære the Point* B, *to which a Tangent* T B *and right Line* P B *being drawn, the Angle* PBT *may be a* Minimum *or the leaſt poſſible?*



SOLUTION.

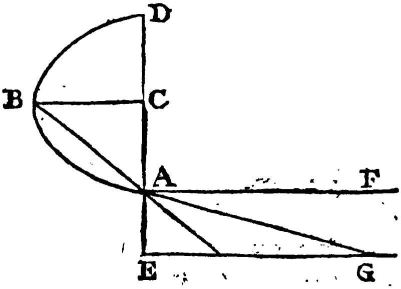
Draw the Radius C B. Now, becauſe the Tangent to a Circle is always perpendicular to the Radius, it is evident that the Angle PBT will be the leaſt poſſible, when the Angle PBC is the greateſt: But, becauſe CB : Sine ∠ BPC : : CP : Sine ∠ PBC; therefore the Angle PBC will be greateſt; when the Angle CPB is a right one: Therefore the Angle PBT will be the leaſt poſſible when PB is perpendicular to CA. *Q. E. I.*

CHAP. XV.

QUESTIONS.

I. THERE is a Door 6 Feet high, being the Opening of a long Paſſage againſt a Street 8 Feet wide. Quære the longeſt Ladder that can be put in at that Door?

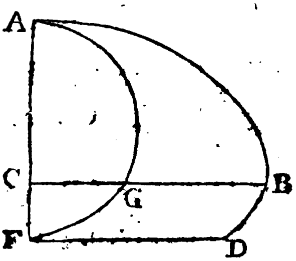
II. Let the Line AG, ſituated as in the *Fig.* between the Parallels AF and EG, be given $=a$; as alſo the Diſtance AE$=b$; then let the End G of the ſaid Line AG, advance towards E, whilſt the other End of the ſaid Line is elevated as much as the Point A will give leave, till the ſaid Line comes to ſtand perpendicular on the Line EG from E to D:



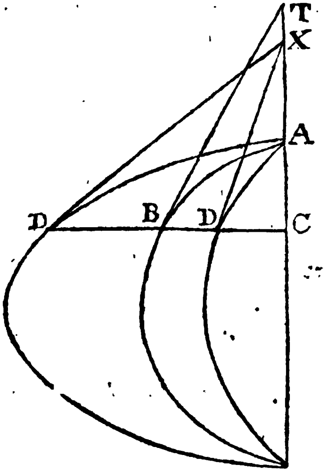
By this Motion 'tis plain that the upper End of the Line AG will deſcribe the Curve ABD: Quære the Nature of the ſaid Curve, as alſo B C its greateſt Diſtance from the Line A D?

III. If a Ball, whoſe Diameter is 4 Inches, be thrown into a conical Glaſs, $\frac{1}{2}$ full of Water, whoſe Diameter is 5 Inches and Altitude 6; how much of the Ball will be immers'd in the Water?

IV. Let A B D be a given curtate Semicycloid, *i. e.* a Semicycloid the Circumference of whoſe generating Semicircle AGF is greater than its Baſe FD. Quære the Point C, in the Diameter A F, where the Ordinate C B is a *Maximum* or the greateſt poſſible?

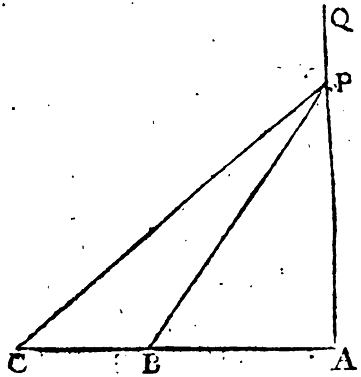


V. Let AB be any given geometrical Curve, and TB a Tangent to it at any Point B: Then if another Curve AD, be ſo drawn, as that its Ordinate DC, ſhall always be in a given Ratio to the correſponding Segment of the former Curve AB; I ſay, that BT will be to TC, as the



the correſponding Segment of the Curve A B is to the Subtangent C X. Quære the Demonſtration?

VI. Let the Diſtances AB and AC from the Perpendicular AQ be given. Quære the Point P where the Angle B P C will be the greateſt poſſible?



FINIS.